Calculus Tricks #2

This set of calculus tricks explains the chain rule and the product-quotient rule. For the purposes of this course, our only need for these rules will be to show that:

- The percentage change in a product of two variables is equal to the sum of the percentage changes in each of the two variables.
- The percentage change in the ratio of two variables is equal to the percentage change in the numerator minus the percentage change in the denominator.

For example, if we're interested in the percentage change in Total Revenue, i.e. $TR = p \cdot Q$, then:

$$\frac{\Delta TR}{TR} = \frac{\Delta(p \cdot Q)}{(p \cdot Q)} = \frac{\Delta p}{p} + \frac{\Delta Q}{Q}$$

To take another example, if we're interested in the percentage change in GDP per capita, i.e. GDP/N (where N denotes population), then:

$$\frac{\Delta \text{GDP per capita}}{\text{GDP per capita}} = \frac{\Delta(GDP/N)}{(GDP/N)} = \frac{\Delta GDP}{GDP} - \frac{\Delta N}{N}$$

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the chain rule

Say you are considering a function that is a function of a function. That is:

$$h(x) = f(g(x))$$

In other words, the value of $h(x)$ changes as the function named "f" changes and the function named "f" changes as the function $g(x)$ changes.

To analyze this change, we can analyze a chain of causality that runs from x to $h(x)$.

$$x \rightarrow g(x) \rightarrow f(g(x)) = h(x)$$

So the derivative of $h(x)$ with respect to x is:

$$\frac{d\,h(x)}{d\,x} = \frac{d\,f(x)}{d\,g(x)} \cdot \frac{d\,g(x)}{d\,x}$$

which looks like the chain of causality flipped around:

$$h(x) = f(g(x)) \leftarrow g(x) \leftarrow x$$

So for example, if $g(x) = 3x + 1$ and if $f(g(x)) = (g(x))^2$, then $h(x) = (3x+1)^2$.

So there are two ways to take the derivative of $h(x)$ with respect to x. Using the methods you already learned, you could expand the terms in the function $h(x)$:

$$h(x) = (3x+1)^2 = 9x^2 + 6x + 1$$

and then take the derivative of $h(x)$ with respect to x, so that:

$$h'(x) = \frac{d\,h(x)}{d\,x} = 18x + 6$$

Expanding the terms of $(3x+1)^2$ can be rather tedious when you're working with a complicated function. Fortunately, the chain rule enables us to arrive at the same result, but in a somewhat quicker fashion:

$$\left.\begin{array}{lcr} f(g(x)) = (g(x))^2 & \Rightarrow & f'(g(x)) = 2 \cdot g(x) \\ \\ g(x) = 3x+1 & \Rightarrow & g'(x) = 3 \end{array}\right\} \rightarrow \begin{array}{rl} h'(x) &= f'(g(x)) \cdot g'(x) \\ &= 2 \cdot g(x) \cdot 3 \\ &= 6 \cdot (3x+1) \\ &= 18x + 6 \end{array}$$

which yields exactly the same result as the one above.

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the product-quotient rule

Say you are considering a function that is the product of two functions, each of which is a function of the variable x. That is:

$$h(x) = f(x) \cdot g(x)$$

If we knew the explicit functional forms of $f(x)$ and $g(x)$, then we could multiply $f(x)$ by $g(x)$ and take the derivative of $h(x)$ with respect to x using the rules you already know. For example,

$$\text{if } f(x) = 3x \text{ and } g(x) = x^2, \text{ then } \begin{array}{rl} h(x) &= f(x) \cdot g(x) \\ &= 3x \cdot x^2 \\ &= 3x^3 \end{array} \text{ and } \frac{d\,h(x)}{d\,x} = h'(x) = \mathbf{9x^2}$$

But we can also consider the change in $h(x)$ as $f(x)$ changes holding $g(x)$ constant and the change in $h(x)$ as $g(x)$ changes holding $f(x)$ constant.

In other words: $$\frac{d\,h(x)}{d\,x} = \frac{d\,f(x)}{d\,x} \cdot g(x) + \frac{d\,g(x)}{d\,x} \cdot f(x) \quad \text{or} \quad h'(x) = f'(x) \cdot g(x) + g'(x) \cdot f(x)$$

Using the previous case where $f(x) = 3x$ and $g(x) = x^2$, we can write:

$$\begin{array}{rl} h'(x) &= f'(x) \cdot g(x) + g'(x) \cdot f(x) \\ &= 3 \cdot x^2 + 2x \cdot 3x \\ &= 3x^2 + 6x^2 \\ &= 9x^2 \end{array}$$

which yields exactly the same result as the one above.

Now let's say you are considering a function that is a ratio of two functions, each of which is a function of the variable x. That is:

$$h(x)=\frac{f(x)}{g(x)} \quad \text{which can be rewritten as:} \quad h(x)=f(x)\cdot(g(x))^{-1}$$

To find the derivative of $h(x)$ with respect to x, we can perform the exact same analysis as we did in the previous example, but with the twist that we also have to use the chain rule on the term $(g(x))^{-1}$.

If we define a function $k(x)$ which is identically equal to $(g(x))^{-1}$, i.e. $k(x)\equiv(g(x))^{-1}$, then we can rewrite the function $h(x)$ as:

$$h(x)=f(x)\cdot k(x)$$

The derivative of $h(x)$ with respect to x is:

$$h'(x)=f'(x)\cdot k(x)+k'(x)\cdot f(x)$$

And the derivative of $k(x)$ with respect to x is:

$$\frac{d\,k(x)}{d\,x}=\frac{d\,(g(x))^{-1}}{d\,g(x)}\cdot\frac{d\,g(x)}{d\,x}$$

$$k'(x)=-1\cdot(g(x))^{-2}\cdot g'(x)=-\frac{g'(x)}{(g(x))^2}$$

Plugging that into the derivative of $h(x)$ with respect to x:

$$h'(x)=f'(x)\cdot(g(x))^{-1}-\frac{g'(x)}{(g(x))^2}\cdot f(x)$$

$$h'(x)=\frac{f'(x)}{g(x)}-\frac{g'(x)\cdot f(x)}{(g(x))^2}$$

So let's consider: $h(x)=\frac{f(x)}{g(x)}$, where $f(x)=6x^4+2x^2$ and $g(x)=2x$. In such a case, $h(x)=3x^3+x$ and $h'(x)=\mathbf{9x^2+1}$. To illustrate the rule we just derived, let's use the rule to obtain the same result:

$$h'(x)=\frac{f'(x)}{g(x)}-\frac{g'(x)\cdot f(x)}{(g(x))^2}$$

$$\left.\begin{array}{lll} f(x)=6x\ \ +2x^2 & \Rightarrow & f'(x)=24x^3+4x \\ g(x)=2x & \Rightarrow & g'(x)=2 \end{array}\right\} \Rightarrow \quad h'(x)=\frac{24x^3+4x}{2x}-\frac{2\cdot(6x^4+2x^2)}{(2x)^2}$$

$$h'(x)=12x^2+2\ \ -3x^2-1 \qquad =\mathbf{9x^2+1}$$

Now, let's return to the original purpose of this set of Calculus Tricks, i.e. to show that:

- The percentage change in a product of two variables is equal to the sum of the percentage changes in each of the two variables.
- The percentage change in the ratio of two variables is equal to the percentage change in the numerator minus the percentage change in the denominator.

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Example #1 – a percentage change in Total Revenue

Once again Total Revenue is given by $TR = p \cdot Q$. Let's assume now that the price of output and the quantity of output produced evolve over time, so that $p = p(t)$ and $Q = Q(t)$, where "t" represents time. In such a case Total Revenue would also evolve over time $TR = TR(t)$.

So what's the percentage change in Total Revenue over time? First, we need to find the changes:

$$\frac{d\,TR(t)}{d\,t} = \frac{d\,p(t)\cdot Q(t)}{d\,t} = \frac{d\,p(t)}{d\,t}\cdot Q(t) + p(t)\cdot\frac{d\,Q(t)}{d\,t}$$

$$= TR'(t) \qquad = p'(t)\cdot Q(t) + p(t)\cdot Q'(t)$$

Since we're interested in a percentage change, we need to divide both sides by Total Revenue to get the percentage change in Total Revenue:

$$\frac{TR'(t)}{TR(t)} = \frac{p'(t)\cdot Q(t)}{p(t)\cdot Q(t)} + \frac{p(t)\cdot Q'(t)}{p(t)\cdot Q(t)}$$

$$\underset{TR}{\%\Delta} = \frac{p'(t)}{p(t)} + \frac{Q'(t)}{Q(t)} = \underset{\text{price}}{\%\Delta} + \underset{\text{quantity}}{\%\Delta}$$

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a note on time derivatives

When working with dynamic changes – that is: a change over time – economists usually denote a time derivative by placing a dot over the variable. I will frequently use this notation.

So for example, the derivative of price with respect to time would be denoted by $\dot{p}$

$$\frac{d\,p(t)}{d\,t} = p'(t) = \dot{p}$$

and the derivative of quantity with respect to time would be denoted by $\dot{Q}$

$$\frac{d\,Q(t)}{d\,t} = Q'(t) = \dot{Q}$$

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Example #2 – a percentage change in the Capital-Labor ratio

The Capital-Labor ratio – denoted: k – is defined as: $k \equiv \frac{K}{L}$, where K and L denotes capital and labor respectively.

Suppose that these two variables evolve over time so that: $K = K(t)$ and $L = L(t)$. This implies that the Capital-Labor ratio also evolves over time, so $k = k(t)$.

To avoid clutter, I'll drop the "t" from the functional notations.

So how does the Capital-Labor ratio evolve over time?

$$\frac{dk}{dt} = \frac{d}{dt}\cdot\left(K \cdot L^{-1}\right)$$

$$\dot{k} = L^{-1}\cdot\frac{dK}{dt} + K\cdot\frac{dL^{-1}}{dL}\cdot\frac{dL}{dt}$$

$$= \frac{\dot{K}}{L} - K\cdot\frac{\dot{L}}{L^2} = \frac{K}{L}\cdot\left(\frac{\dot{K}}{K}-\frac{\dot{L}}{L}\right)$$

Since $k \equiv \frac{K}{L}$, the derivation above implies that:

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}$$

The percentage change in the Capital-Labor ratio over time is equal to the percentage change in Capital over time minus the percentage change in Labor over time.

Some students have told me that they understand the product-quotient rule better when I explain the rules using difference equations.

Example #1 revisited – a percentage change in Total Revenue

Since Total Revenue is given by: $TR = p \cdot Q$, the percentage change in Total Revenue is:

$$\frac{\Delta TR}{TR} = \frac{\Delta(p \cdot Q)}{p \cdot Q} = \frac{p_2 \cdot Q_2 - p_1 \cdot Q_1}{p_1 \cdot Q_1}$$

where: p_1 is the inital price, Q_1 is the inital quantity, p_2 is the new price, Q_2 is the new quantity

Next, we're going to add a zero to the equation above. Adding zero leaves the value of the percentage change in Total Revenue unchanged.

We're going to add that zero in an unusual manner. The zero that we're going to add is:

$$0 = \frac{p_1 \cdot Q_2 - p_1 \cdot Q_2}{p_1 \cdot Q_1}$$

Adding our "unusual zero" yields:

$$\frac{\Delta TR}{TR} = \frac{p_2 \cdot Q_2 - p_1 \cdot Q_1}{p_1 \cdot Q_1} + \frac{p_1 \cdot Q_2 - p_1 \cdot Q_2}{p_1 \cdot Q_1}$$

Rearranging terms, we get:

$$\frac{\Delta TR}{TR} = \frac{(p_2 - p_1) \cdot Q_2}{p_1 \cdot Q_1} + \frac{p_1 \cdot (Q_2 - Q_1)}{p_1 \cdot Q_1}$$

Now notice that: $\Delta p = (p_2 - p_1)$ and $\Delta Q = (Q_2 - Q_1)$, therefore:

$$\frac{\Delta TR}{TR} = \frac{\Delta p}{p_1} \cdot \frac{Q_2}{Q_1} + \frac{\Delta Q}{Q_1}$$

Since we're considering very small changes: $\Delta Q \approx 0$, which implies that: $Q_2 \approx Q_1$ and $\frac{Q_2}{Q_1} \approx 1$.

Therefore we can write:

$$\frac{\Delta TR}{TR} = \frac{\Delta p}{p} + \frac{\Delta Q}{Q}$$

Example #2 revisited – a percentage change in the Capital-Labor ratio

Once again, define k as the Capital-Labor ratio, i.e.: $k \equiv \frac{K}{L}$, where K denote capital and L denotes labor. The percentage change in the Capital-Labor ratio is:

$$\frac{\Delta k}{k} = \frac{\Delta(K/L)}{K/L} = \frac{\frac{K_2}{L_2} - \frac{K_1}{L_1}}{K_1/L_1}$$

where: K_1 is the inital capital stock, L_1 is the inital labor force; K_2 is the new capital stock, L_2 is the new labor force

Once again, we're going to add an "unusual zero."

$$0 = \frac{\frac{K_1}{L_2} - \frac{K_1}{L_2}}{K_1/L_1}$$

Adding our "unusual zero" yields:

$$\frac{\Delta(K/L)}{K/L} = \frac{\frac{K_2}{L_2} - \frac{K_1}{L_1}}{K_1/L_1} + \frac{\frac{K_1}{L_2} - \frac{K_1}{L_2}}{K_1/L_1}$$

Rearranging terms, we get:

$$\frac{\Delta(K/L)}{K/L} = \frac{\frac{K_2}{L_2} - \frac{K_1}{L_2}}{K_1/L_1} + \frac{\frac{K_1}{L_2} - \frac{K_1}{L_1}}{K_1/L_1}$$

$$= \left(\frac{K_2}{L_2} - \frac{K_1}{L_2}\right) \cdot \frac{L_1}{K_1} + \left(\frac{K_1}{L_2} - \frac{K_1}{L_1}\right) \cdot \frac{L_1}{K_1}$$

$$= \left(\frac{K_2 - K_1}{K_1}\right) \cdot \frac{L_1}{L_2} + \left(\frac{L_1}{L_2} - \frac{L_1}{L_1}\right) \cdot \frac{K_1}{K_1}$$

$$= \frac{\Delta K}{K_1} \cdot \frac{L_1}{L_2} + \left(\frac{L_1}{L_2} - 1\right) \quad \leftarrow \text{note that: } 1 = \frac{L_2}{L_2}$$

$$= \frac{\Delta K}{K_1} \cdot \frac{L_1}{L_2} + \left(\frac{L_1 - L_2}{L_2}\right)$$

$$= \frac{\Delta K}{K_1} \cdot \frac{L_1}{L_2} - \frac{\Delta L}{L_2}$$

The derivation above uses the definitions: $\Delta K = K_2 - K_1$ and $\Delta L = L_2 - L_1$.

Since we're considering very small changes: $\Delta L \approx 0$, which implies that: $L_2 \approx L_1$ and $\frac{L_1}{L_2} \approx 1$.

Therefore we can write:

$$\frac{\Delta(K/L)}{K/L} = \frac{\Delta K}{K} - \frac{\Delta L}{L}$$