Lecture 4

Economic Growth: the Solow Model

Eric Doviak

Economic Growth and Economic Fluctuations

Why Study Economic Growth?

Is there some action a government of India could take that would lead the Indian economy to grow like Indonesia's or Egypt's? If so, what, exactly? If not, what is it about the "nature of India" that makes it so? The consequences for human welfare involved in questions like these are simply staggering: Once one starts to think about them, it is hard to think about anything else.

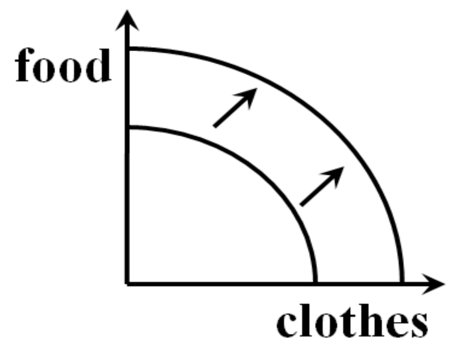
– Robert E. Lucas, Jr.

What is Economic Growth?

- **Before the Industrial Revolution in Great Britain, every society in the world was agrarian.**
- **Then technical change and capital accumulation increased British society's ability to produce textiles and agricultural products**
- **and a rapid and sustained increase in real output per capita began**
- **As a result, more could be produced with fewer resources**
  - new products
  - more output and
  - wider choice

- **Economic growth shifts the society's production possibility frontier up and to the right**

- **Economic growth allows each member of society to produce and consume more of all goods**



Plan of this Lecture

- **In the previous lecture, we learned some basic measures of how national income is distributed among factors of production and how national income is allocated among the goods produced.**
- **In this lecture, we'll examine the model of economic growth developed by Robert M. Solow in the 1950s.**

- **The Solow Model was one of the first attempts to describe how:**
  - **saving,**
  - **population growth and**
  - **technological progress**
- **affect the growth of output per worker over time – i.e. we're looking at LONG RUN economic growth.**

- **We'll use Solow's model to examine:**
  - **why the standards of living vary so widely among countries and**
  - **how economic policy can be used to influence standards of living**
  - **How much of an economy's output should be consumed today and how much should be saved for the future?**

Demand-Side Assumptions

- **To simplify the discussion, we'll examine a closed economy without a government. In other words:**
  - **there is no international trade** (so net exports equal zero) **and**
  - **government purchases equal zero**
- **so output is divided among consumption and investment:**

$$Y = C + I$$

- **Ultimately, we want to examine living standards, so we want to focus on per capita output, consumption and investment.**
- **It's easier to examine per worker variables in this model however.**

- **Nonetheless, per worker variables will yield fairly good approximations of living standards, so if we denote the labor force by L, we can define:**
  - **output per worker** as: $y \equiv Y/L$
  - **consumption per worker** as: $c \equiv C/L$ and **so:** $y = c + i$
  - **investment per worker** as: $i \equiv I/L$

Demand-Side Assumptions

- **Consumption per worker is the amount of output that is not invested**

$$c = y - i$$

- **The Solow Model assumes that consumption and investment** (per worker) **are proportional to income:**

$$c = (1 - s) \cdot y \qquad \textbf{where: } i = s \cdot y$$

- **So that the saving rate – denoted by the letter s – is constant.**

- **In other words, every year a fraction $(1-s)$ of income is consumed and a fraction s of income is saved.**

Supply-Side Assumptions

- **We'll also assume that:**
  - **output is produced using capital,** K**, and labor,** L
  - **there is no technological progress** (we'll drop this assumption later)
  - **the production function exhibits constant returns to scale (CRS)**

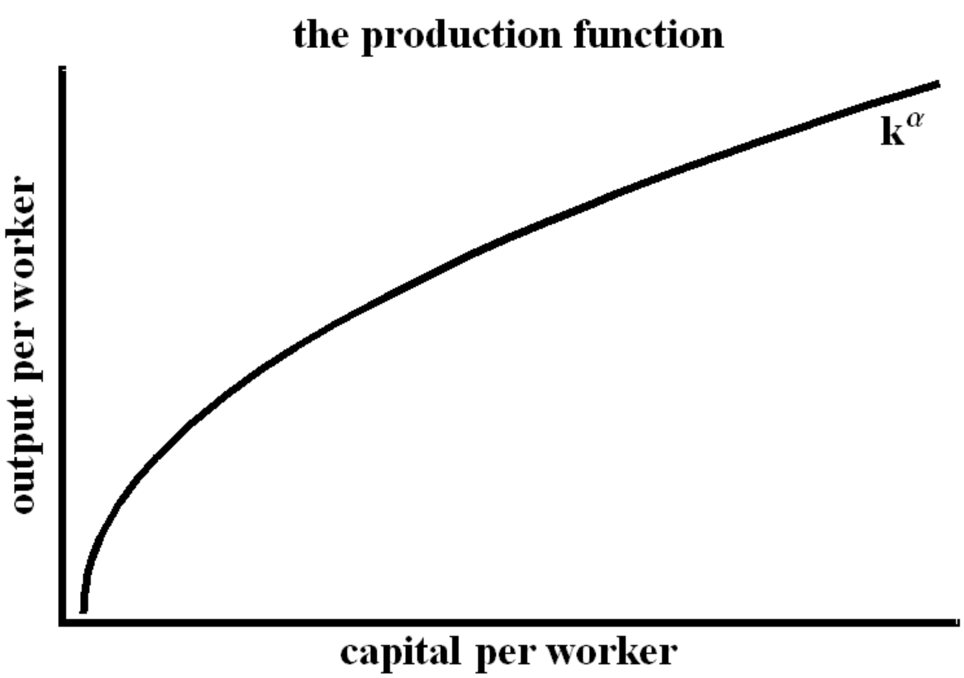
$$Y = K^{\alpha} \cdot L^{1-\alpha} \qquad \text{where: } 0 < \alpha < 1$$

- **Once again, we want to focus on per worker variables, so define:**
  - **output per worker** as: $y \equiv Y/L$ and
  - **capital per worker** as: $k \equiv K/L$

- **One convenient feature of the assumption of CRS is that we can define output per worker entirely in terms of capital per worker:**

$$\left.\begin{aligned} \frac{Y}{L} &= K^{\alpha} \cdot \frac{L^{1-\alpha}}{L} \\ &= K^{\alpha} \cdot L^{-\alpha} \end{aligned}\right| \quad \Rightarrow \quad y = k^{\alpha}$$

Production per Worker



Supply-Side Assumptions

- **Another convenient feature of the assumption of CRS concerns the Marginal Product of Capital (MPK)** – the derivative of output per worker with respect to capital**:**

$$MPK \equiv \frac{dY}{dK}$$

- **the Marginal Product of Capital equals the derivative of output per worker with respect to capital per worker:**

$$\frac{dY}{dK} = \alpha \cdot K^{\alpha-1} \cdot L^{1-\alpha} \qquad y = k^{\alpha}$$

$$\frac{dY}{dK} = \alpha \cdot \frac{K^{\alpha-1}}{L^{\alpha-1}} = \alpha \cdot k^{\alpha-1} \qquad \frac{dy}{dk} = \alpha \cdot k^{\alpha-1}$$

- **What this tells us is that increases in the capital stock per worker increase output per worker, but each successive increase in the capital stock yield ever smaller increases in output per worker – because output per worker exhibits diminishing marginal returns.**

Accumulation of Capital

- **The underlying theory behind the Solow Model:**
  - **countries with higher levels of capital per worker**
  - **have higher levels of output per worker.**

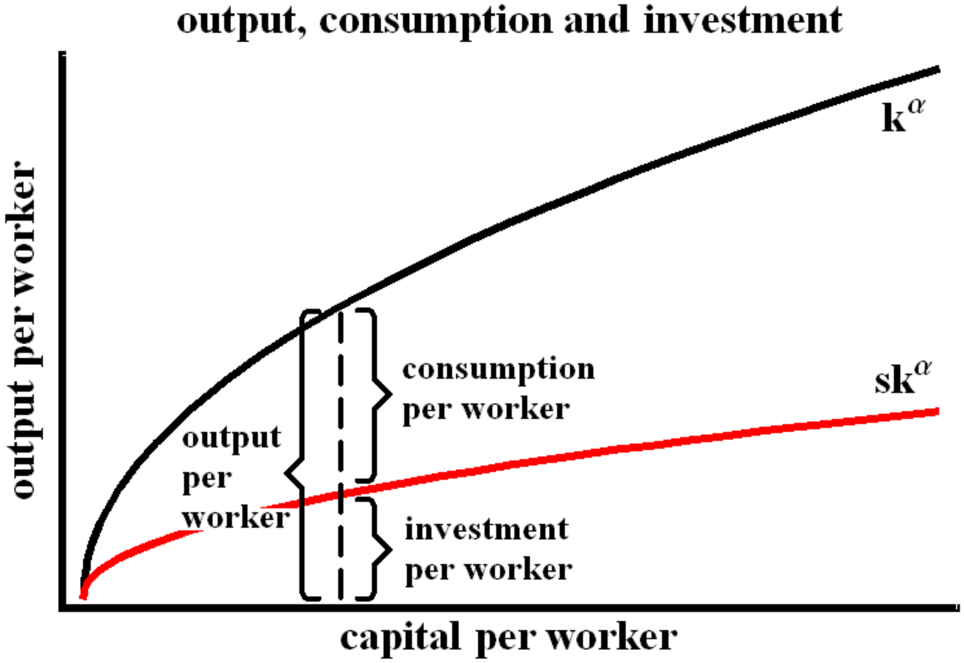
- **Think about that a second.**
- **If Solow's theory is correct, then all we have to do to increase output per worker – and lift billions of people out of poverty – is increase the amount of capital that they have to work with.**

- **So what determines the level of capital per worker in a country?**
- **The Solow Model assumes that:**
  - **investment increases the capital stock, but**
  - **a constant fraction of the capital stock depreciates each year**

- **Our definition of investment,** $i = s \cdot y$ **implies that annual investment in capital is a fraction,** s**, of the total output per year, i.e.** $I = s \cdot Y$
- **Let** δ **denote the fraction of the capital stock that depreciates in a year.**

- **Therefore:** $\dot{K} = sY - \delta K$ **where:** $\dot{K} = \frac{dK}{dt}$

Tradeoff between Consumption and Investment



- **The saving rate s determines the allocation of output per worker between consumption and investment.**

Evolution of Capital per Worker

- **Now that we now how the total capital stock evolves from year to year, finding out how the capital stock per worker evolves from one year to the next is straightforward.**
- Recall from the Calculus Tricks that the percentage change in a ratio is equal to the percentage change in the numerator minus the percentage change in the denominator.
- **So we can find the evolution capital per worker over time:**

$$\frac{\dot{k}}{k} = \left(\frac{\dot{K}}{K} - \frac{\dot{L}}{L}\right) \Rightarrow \dot{k} = \frac{K}{L}\cdot\left(\frac{sY - \delta K}{K} - \frac{\dot{L}}{L}\right)$$

$$= \frac{sY - \delta K}{L} - k \cdot \frac{\dot{L}}{L}$$

$$\text{define: } n \equiv \frac{\dot{L}}{L} \qquad \mathbf{\dot{k} = sk^{\alpha} - (\delta + n) \cdot k}$$

- Note that: n is the constant **exogenous** annual growth rate of the labor force
- the model assumes that there are no cyclical fluctuations in employment

Key Equation of the Solow Model

$$\dot{\mathbf{k}} = \mathbf{sk}^{\alpha} - (\boldsymbol{\delta} + \mathbf{n}) \cdot \mathbf{k}$$

- **Growth of the capital stock per worker over time,** $\dot{k}$
  - **is an increasing function of investment per worker, i.e.** sk^{α}
  - **a decreasing function of the depreciation rate** and
  - **a decreasing function of the growth rate of the labor force**

- **Although we've used math to obtain this result, the result should also be intuitive:**
  - **The capital stock per worker increases at higher saving rates because at higher saving rates more output is being devoted to accumulating capital.**
  - **By definition, depreciation decreases the capital stock, so faster rates of depreciation reduce the capital stock per worker.**
  - **Faster rates of growth of the labor force will also lead to lower levels of capital per worker, because the total capital stock must be spread over a larger labor force.**

Evolution of Capital per Worker

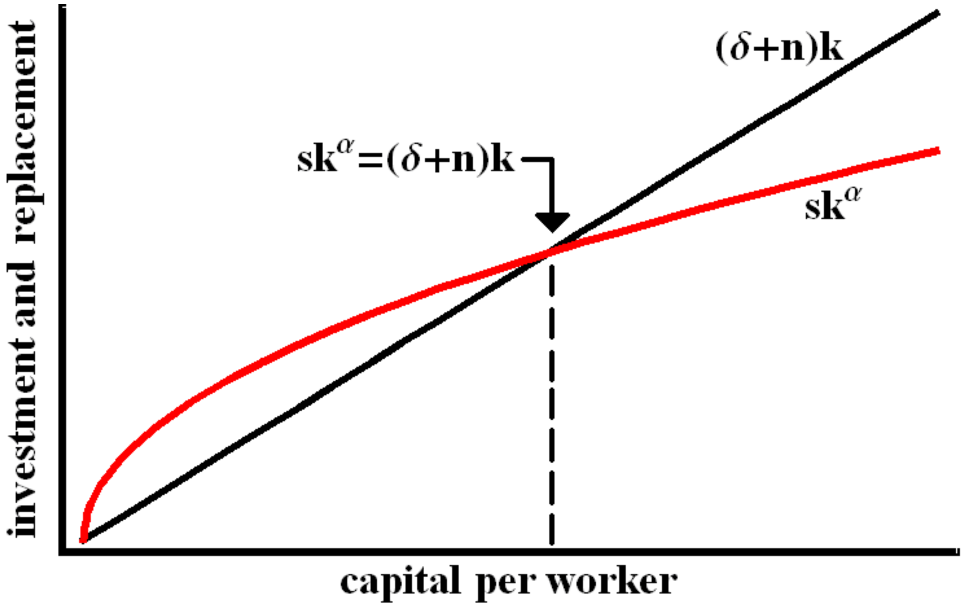
Whether capital per worker is growing, falling or remaining constant over time, depends on whether investment in new capital per worker exceeds, falls short of or is equal to the replacement requirement: $(n+\delta)\cdot k$**.**

- **if** $sk^{\alpha} > (n+\delta)\cdot k$**, then capital per worker increases over time**
  - in this case, **investment in new capital per worker exceeds the replacement requirement** and
  - **output per worker is growing over time**

- **if** $sk^{\alpha} < (n+\delta)\cdot k$**, then capital per worker decreases over time**
  - in this case, **investment in new capital per worker falls short of the replacement requirement** and
  - **output per worker is falling over time**

- **if** $sk^{\alpha} = (n+\delta)\cdot k$**, then capital per worker remains constant over time**
  - in this case, **investment in new capital per worker equals the replacement requirement** and
  - **output per worker is constant over time**
  - **this is called the steady state** (since the level of capital per worker is "steady")

Steady State



- **The capital per worker must converge to the steady state**
- **Once capital per worker converges to the steady state level it remains at that level,** unless the saving rate, depreciation rate or growth rate of the labor force changes.

Convergence to the Steady State

- **As an example of convergence consider an economy that initially starts at a level of capital per worker that is below the steady state level.**
- **If initial** $k = 1$ **and if** $\alpha = 0.5$ $y = k^{0.5}$ $s = 0.08$ $\delta = 0.02$ and $n = 0.02$

year	k	y	c=(1–s)y	$i = s \cdot k^{\alpha}$	$(\delta + n) \cdot k$	Δk
0	1.000	1.000	0.920	0.080	0.040	0.0400
1	1.040	1.020	0.938	0.082	0.042	0.0400
2	1.080	1.039	0.956	0.083	0.043	0.0399
3	1.120	1.058	0.974	0.085	0.045	0.0399
4	1.160	1.077	0.991	0.086	0.046	0.0398
5	1.200	1.095	1.008	0.088	0.048	0.0396
…						
10	1.396	1.182	1.087	0.095	0.056	0.0387
…						
25	1.945	1.394	1.283	0.112	0.078	0.0338
…						
100	3.484	1.867	1.717	0.149	0.139	0.0100
…						
∞	4.000	2.000	1.840	0.160	0.160	0

- After 35 years, this economy's level of output per worker will have converged halfway to its steady state level – i.e. $y = 1.5$ at 35 years.

Steady State

- **Once the economy has converged to its steady state, the level of capital per worker stops growing** (or falling as the case may be)**, i.e.** $\dot{k} = 0$
- **At the steady state:** $sk^{\alpha} = (\delta + n) \cdot k$**. If we solve this equation for** k**, we find the steady state level of capital per worker:**

$$k_{SS} = \left(\frac{s}{n + \delta} \right)^{\frac{1}{1-\alpha}}$$

- **which implies that the steady state level of output per worker is:**

$$y_{SS} = \left(\frac{s}{n + \delta} \right)^{\frac{\alpha}{1-\alpha}}$$

- **In the long-run, the steady-state level of output per worker is constant and depends only on:**
  - **the saving rate**
  - **the labor force growth rate and**
  - **the rate at which capital depreciates**

Economic Growth

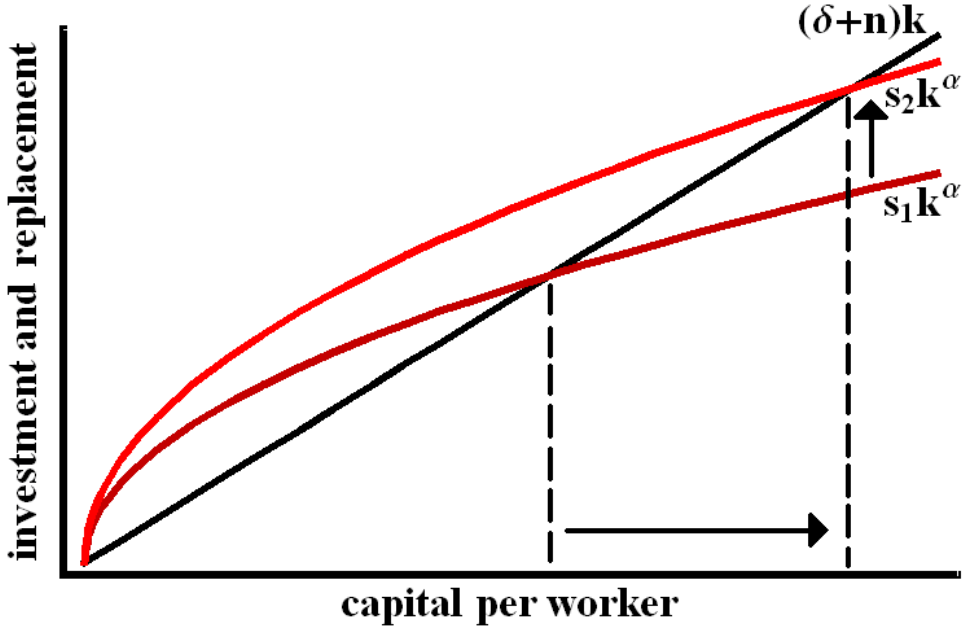
- **The interesting thing about this model of economic growth is that there is no growth once the economy reaches steady state.**
- **But what do politicians say?**
  - Politicians say tax cuts will be great for economic growth
  - Politicians say protecting open space will be great for economic growth
  - Politicians say building a new stadium will be great for economic growth
- **Are they lying?**

- **Public policy cannot affect the steady state growth rate.**
- **But public policy can affect the steady state level of output per worker, which will affect living standards.**

- **If policymakers found a way to increase the saving rate the economy will converge to a higher steady state level of output per worker.**

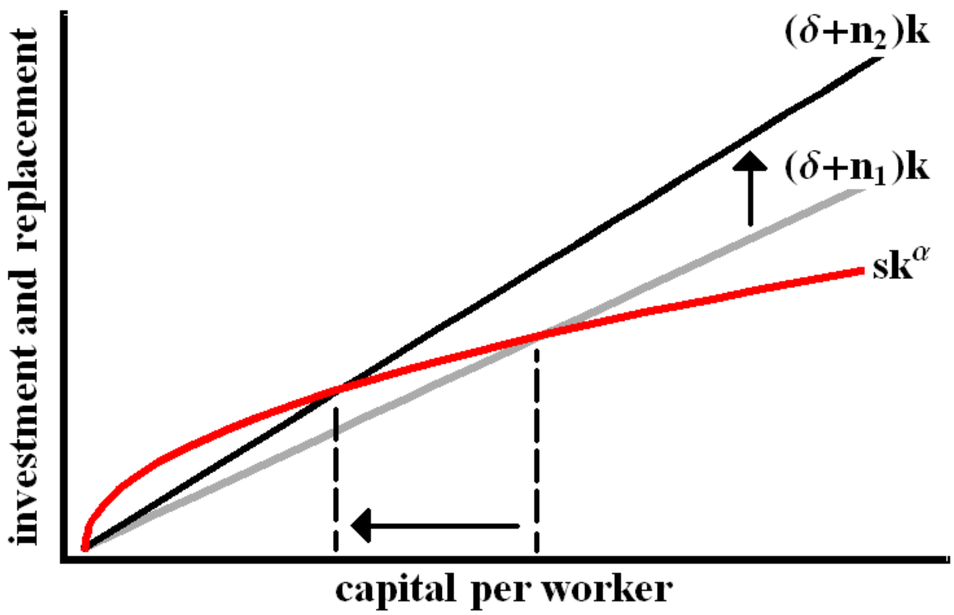
- **Conversely, if policymakers pursued a policy that increased the labor force growth rate, then the economy would converge to a lower steady state level of output per worker.**

Increasing the Saving Rate



- **Many economists favor low corporate tax rates as a way to encourage saving, in the hope that lower rates will stimulate savings/investment.**
- **At a higher saving rate, the economy will converge to a higher steady state level of output per worker.**

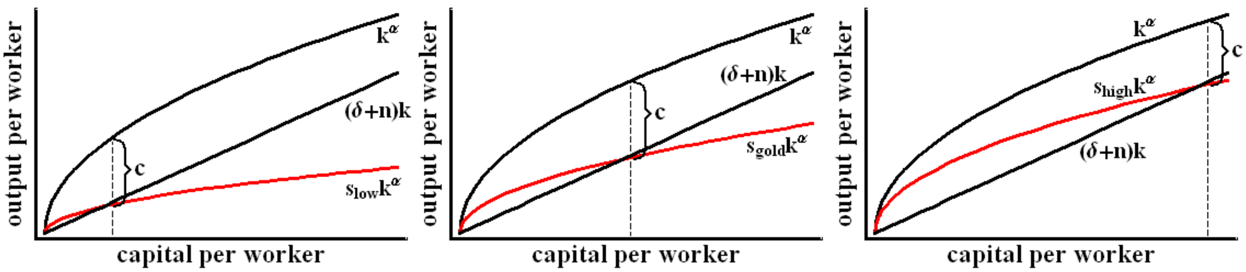
Increasing the Labor Force Growth Rate



- **One reason living standards are low in some countries is because they have high rates of population growth** (high rates of labor force growth)**.**
- **At a higher labor force growth rate, the economy will converge to a lower steady state level of output per worker.**

the Golden Rule

- **So far, it would appear that the goal of public policy should be to reach a higher steady state level of output per worker**
- **In practice, that should be the goal – our saving rate is too low – but …**
- **If a benevolent policymaker could choose the saving rate – which would enable him/her to choose the steady state level of output per worker, then which steady steady state should he/she choose?**
  - Extreme example #1: You wouldn't want a saving rate of 1%
  - Extreme example #2: You wouldn't want a saving rate of 99%



- **If the policymaker followed the Golden Rule of "Do unto others …" then he/she would want to choose the steady state with the highest level of consumption per worker.** This case is depicted in the middle panel.

the Golden Rule

- Using our Calculus Tricks, we can find the saving rate which maximizes consumption across steady states in the same way that we found a firm's maximum profit:
  - take the derivative of consumption with respect to the saving rate
  - and set it equal to zero
- This time it's easier because we can use the chain rule:

$$\underset{s}{\text{Max}} \quad c_{SS}(s) = k_{SS}(s)^{\alpha} - (n+\delta)\cdot k_{SS}(s)$$

$$c'_{SS}(s) = \left(\alpha \cdot k_{SS}^{\alpha-1} - (n+\delta)\right)\cdot k'_{SS}(s) = 0 \quad \Rightarrow \quad \alpha \cdot k_{GOLD}^{\alpha-1} = (n+\delta)$$

- In other words, the Golden Rule steady state level of capital per worker corresponds to level of capital per worker which equates the Marginal Product of Capital to $(n+\delta)$.

- **The Golden Rule level is a CHOICE.**
- **The economy does NOT converge to the Golden Rule level on its own!**

Technological Progress

- **You may have noticed that if the steady state level of output per worker is constant, then:**
  - **as the economy approaches steady state**
  - **growth of output per worker is zero and therefore**
  - **growth of income per worker is zero**

- **Is this realistic? No.**

- **We can introduce more realism into the model if we introduce technological progress into the model.**
- Technological progress occurs when firms find ways to produce more from the same amount of resources.
- **If we define a variable A to denote the efficiency of labor**
  - **which reflects society's knowledge about production methods or**
  - **which reflects improvements in the health, education of skills of the labor force**
- **then as the available technology improves, the efficiency of labor rises.**

Technological Progress

- **So redefine the production function as:**

$$Y = K^{\alpha} \cdot (AL)^{1-\alpha} \qquad \text{where: } 0 < \alpha < 1$$

- **Once again, we want to focus on per worker variables, but now we have to focus on labor in efficiency units, so define:**
  - **output per unit of effective labor** as: $\tilde{y} \equiv Y/AL$ and
  - **capital per unit of effective labor** as: $\tilde{k} \equiv K/AL$

- **Since the production function still assumes constant returns to scale, so we can define output per unit of effective labor in terms of capital per unit of effective labor:**

$$\left.\begin{aligned} \frac{Y}{AL} &= K^{\alpha} \cdot \frac{(AL)^{1-\alpha}}{AL} \\ &= K^{\alpha} \cdot (AL)^{-\alpha} \end{aligned}\right\} \Rightarrow \quad \tilde{y} = \tilde{k}^{\alpha}$$

Evolution of Capital per unit of Effective Labor

- **Using the Calculus Tricks once again, we can find the evolution capital per unit of effective labor over time:**

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \left(\frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L}\right) \Rightarrow \dot{\tilde{k}} = \frac{K}{AL} \cdot \left(\frac{sY - \delta K}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L}\right)$$

$$\text{define: } g \equiv \frac{\dot{A}}{A} \qquad = \frac{sY - \delta K}{AL} - \tilde{k} \cdot \left(\frac{\dot{A}}{A} + \frac{\dot{L}}{L}\right)$$

$$\text{define: } n \equiv \frac{\dot{L}}{L} \qquad \dot{\tilde{\mathbf{k}}} = \mathbf{s}\tilde{\mathbf{k}}^{\boldsymbol{\alpha}} - (\boldsymbol{\delta} + \mathbf{g} + \mathbf{n}) \cdot \tilde{\mathbf{k}}$$

- Note that: g is the **exogenous** annual growth rate of technological progress
  n is the **exogenous** annual growth rate of the labor force

- We're assuming that technology grows at a constant rate and that there are no cyclical fluctuations in the level of technology.
- We're still assuming that the labor force grows at a constant rate and that there are no cyclical fluctuations in employment.

Key Equation of the Solow Model with Technological Progress

$$\dot{\tilde{\mathbf{k}}} = \mathbf{s}\tilde{\mathbf{k}}^{\boldsymbol{\alpha}} - (\boldsymbol{\delta} + \mathbf{g} + \mathbf{n}) \cdot \tilde{\mathbf{k}}$$

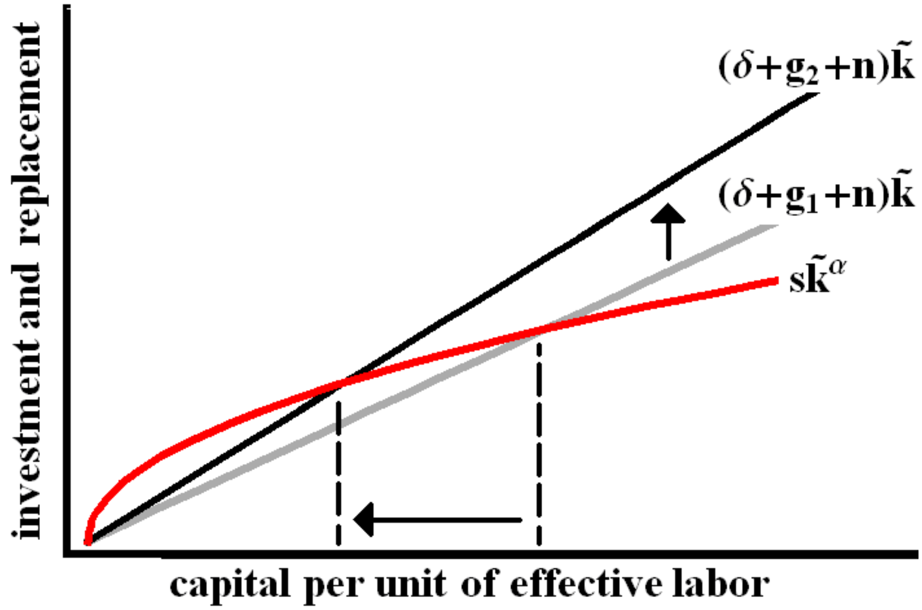
- **This "Key Equation" has the same interpretation as the previous one but this time:**
  - **growth of the capital stock per unit of effective labor over time, $\dot{\tilde{k}}$,**
  - **is a decreasing function of the growth rate of technological progress, g.**
- **And this time the steady state level of the capital stock per unit of effective labor, will occur when:**

$$\dot{\tilde{k}} = 0 \Rightarrow s\tilde{k}^{\alpha} = (\delta + g + n) \cdot \tilde{k}$$

- **Solve this equation for $\tilde{k}$, we can find obtain the steady state levels of capital and output per unit of effective labor:**

$$\tilde{k}_{SS} = \left(\frac{s}{n + g + \delta}\right)^{\frac{1}{1-\alpha}} \quad \textbf{and} \quad \tilde{y}_{SS} = \left(\frac{s}{n + g + \delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Increasing the Rate of Technological Progress



- **If policymakers were able to find a way to increase the rate of technological progress, the steady state level of capital per unit of effective labor would fall, but …**
- **… this would be a GOOD THING**

the Rate of Technological Progress

- **A faster rate of technological progress would lower the steady state level of capital per unit of effective labor, but …**
- **… this would be a GOOD THING**
- **The rate of growth of output per unit of effective labor is:**

$$\frac{\dot{\tilde{y}}}{\tilde{y}} = \frac{\dot{Y}}{Y} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L} = \frac{\dot{Y}}{Y} - g - n$$

- **The rate of growth of output per worker is:**

$$\frac{\dot{y}}{y} = \frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} = \frac{\dot{Y}}{Y} - n$$

- **In steady state, the growth rate of output per unit of effective labor is zero, but this implies that the rate of growth of output per worker is equal to the rate of growth of technological progress.**

$$\frac{\dot{\tilde{y}}}{\tilde{y}} = 0 \quad \Rightarrow \quad \frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} = \frac{\dot{A}}{A} \quad \text{or more simply:} \quad \frac{\dot{y}}{y} = g$$

- **So a faster rate of growth of technological progress implies a rapidly rising standard of living for the residents of that economy**