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Calculus Tricks #1

Calculus is <u>not</u> a pre-requisite for this course. However, the foundations of economics are based on calculus, so what we'll be discussing over the course of the semester is the intuition behind models constructed using calculus.

It's not surprising therefore that the students who do better in economics courses are the ones who have a better understanding of calculus – **even when calculus is not a required part of the course**. So if you want to do well in this course, you should learn a little calculus.

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Many times throughout the course, we'll be discussing marginalism – e.g. marginal cost, marginal revenue, marginal product of labor, marginal product of capital, marginal propensity to consume, marginal propensity to save, etc.

Whenever you see "marginal …" it means "the derivative of …"

A derivative is just a slope. So, for example, let's say **labor is used to produce output**

- if TP stands for Total Production (quantity produced),
- if L stands for Labor input and
- if Δ denotes a change,

then if I write: $\frac{\Delta TP}{\Delta L}$ that's the change in Total Production divided by the change in Labor.

- It's the slope of the total production function.
- It's the derivative of the production function with respect to labor input.
- It's the marginal product of labor (MPL).

So if you understand derivatives, you'll understand the course material much better.

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a few preliminaries – exponents

You should recall from your high school algebra classes that when you see an exponent, it simply means multiply the number by itself the number of times indicated by the exponent.

$$x^3 = x \cdot x \cdot x$$

Now if you divide both sides of the above equation by x:

$$\frac{x^3}{x} = \frac{x \cdot x \cdot x}{x} = x^2$$

But what if you see the something like: x^0? Well, that's simply equal to:

$$x^0 = \frac{x^1}{x} = \frac{x}{x} = 1$$

$$2^3 = 2 \cdot 2 \cdot 2 = 8 = \frac{16}{2} = \frac{2^4}{2}$$

$$2^2 = 2 \cdot 2 = 4 = \frac{8}{2} = \frac{2^3}{2}$$

$$2^1 = 2 = 2 = \frac{4}{2} = \frac{2^2}{2}$$

$$2^0 = 1 = 1 = \frac{2}{2} = \frac{2^1}{2}$$

$$2^{-1} = \frac{1}{2} = \frac{1}{2} = \frac{1}{2} = \frac{2^0}{2}$$

$$2^{-2} = \frac{1}{2 \cdot 2} = \frac{1}{4} = \frac{1/2}{2} = \frac{2^{-1}}{2}$$

$$2^{-3} = \frac{1}{2 \cdot 2 \cdot 2} = \frac{1}{8} = \frac{1/4}{2} = \frac{2^{-2}}{2}$$

Similarly, $x^{-1} = \frac{x^0}{x} = \frac{1}{x}$ and $x^{-2} = \frac{x^{-1}}{x} = \frac{1/x}{x} = \frac{1}{x \cdot x} = \frac{1}{x^2}$

But what about $x^{0.5}$? That's the square root of x: $x^{0.5} = \sqrt{x}$. Ex. $16^{0.5} = \sqrt{16} = 4$

By the same logic as before: $x^{-0.5} = \frac{1}{\sqrt{x}}$. Ex. $9^{-0.5} = \frac{1}{\sqrt{9}} = \frac{1}{3}$

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a few preliminaries – functions

You may have seen something like this in your high school algebra classes: $f(x)$. This notation means that there is a function named "f" whose value depends on the value of the variable called "x."

Some examples of functions in economics include:

- The quantity of output that a firm produces depends on the amount of labor that it employs. In such a case, we can define a function called "TP" (which stands for Total Production) whose value depends on a variable called "L" (which stands for Labor). So we would write: $TP(L)$.
- A firm's total cost of producing output depends on the amount of output that it produces. In such a case, we can define a function called "TC" (which stands for Total Cost) whose value depends on a variable called "Q" (which stands for Quantity). So we would write: $TC(Q)$.
- A firm's total revenue from selling output depends on the amount of output that it produces. In such a case, we can define a function called "TR" (which stands for Total Revenue) whose value depends on a variable called "Q" (which stands for Quantity). So we would write: $TR(Q)$.

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derivatives

Now let's return to the original purpose of these notes – to show you how to take a derivative.

A derivative is the slope of a function. For those of you who saw $f(x)$ in your high school algebra classes, you may recall taking a derivative called "f-prime of x," $f'(x)$.

What you were doing was you were finding the slope of the function $f(x)$. You were finding how much the value of the function $f(x)$ changes as x changes.

So let's define the function: $f(x) = 3x^2$ and let's look at how the value of $f(x)$ changes as we increase x by one unit increments. Once again, let Δ denote a change.

x	$f(x)$	$\frac{\Delta f(x)}{\Delta x}$	true $f'(x)$
0	0	--	0
1	3	3	6
2	12	9	12
3	27	15	18

The third column is our rough measure of the slope. The fourth column – entitled true $f'(x)$ – is the true measure of the slope of $f(x)$ evaluated at each value of x. The values differ greatly between the two columns because we are looking at "large" changes in x (in the third column) as opposed to the infinitesimally small changes described in the notes entitled: "What's the Difference between Marginal Cost and Average Cost?" (The infinitesimally small changes are listed in the fourth column).

Why does it make a difference whether we look at small or large changes? Consider the following derivation of the slope of $f(x)$:

$$f'(x) = \frac{\Delta f(x)}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \frac{3(x + \Delta x)^2 - 3x^2}{\Delta x} = \frac{3(x + \Delta x) \cdot (x + \Delta x) - 3x^2}{\Delta x} = \frac{3\left(x^2 + 2x\Delta x + (\Delta x)^2\right) - 3x^2}{\Delta x}$$

$$= \frac{3x^2 + 6x\Delta x + 3(\Delta x)^2 - 3x^2}{\Delta x} = \frac{6x\Delta x + 3(\Delta x)^2}{\Delta x} = \frac{6x\Delta x}{\Delta x} + \frac{3(\Delta x)^2}{\Delta x}$$

$$f'(x) = 6x + 3\Delta x$$

If we look at one unit changes in the value of x – i.e. $\Delta x = 1$ – then the slope of $f(x)$ evaluated at each value of x is equal to $6x + 3\Delta x$ which equals $6x + 3$ since $\Delta x = 1$.

If we look at changes in x that are so small that the changes are approximately zero – i.e.: $\Delta x \approx 0$ – then the slope of $f(x)$ evaluated at each value of x is approximately equal to $6x$ and gets closer and closer to $6x$ as the change in x goes to zero.

So if $f(x) = 3x^2$, then $f'(x) = 6x$.

Since we'll be looking at infinitesimally small changes in x, we'll stop using the symbol Δ to denote a change and start using the letter d to denote an infinitesimally small change.

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calculus tricks – an easy way to find derivatives

For the purposes of this course, there are only a handful of calculus rules you'll need to know:

1. the constant-function rule
2. the power-function rule,
3. the sum-difference rule,
4. the product-quotient rule and
5. the chain rule.

We'll focus on the first three of these rules now. We'll discuss the last two after we have a firm grasp on the first three.

the constant-function rule

If $f(x) = 3$, then the value of $f(x)$ doesn't change x as changes – i.e. $f(x)$ is constant and equal to 3.

So what's the slope? Zero. Why? Because a change in the value of x doesn't change the value of $f(x)$. In other words, the change the value of $f(x)$ is zero. So if $f(x) = 3$, then $\frac{d\,f(x)}{d\,x} = f'(x) = 0$.

the power-function rule

Now if the value of x in the function $f(x)$ is raised to a power (i.e. it has an exponent), then all we have to do to find the derivative is "roll the exponent over."

To roll the exponent over, multiply the original function by the original exponent and subtract one from the original exponent. For example:

$$f(x) = 5x^3$$

$$\frac{d\,f(x)}{d\,x} = f'(x) = 15x^2$$

$$5x^3 \rightarrow 3 \cdot 5x^{3-1} = 15x^2$$

$$g(x) = 4x^{1/2} = 4\sqrt{x}$$

$$\frac{d\,g(x)}{d\,x} = g'(x) = 2x^{-1/2} = \frac{2}{\sqrt{x}}$$

$$4x^{1/2} \rightarrow \frac{1}{2} \cdot 4x^{\frac{1}{2}-1} = 2x^{-1/2}$$

the sum-difference rule

Now, say the function you are considering contains the variable x in two or more terms.

$$k(x) = 2x^2 - 3x + 5$$

if we define:

$$f(x) = 2x^2 \qquad g(x) = -3x^1 = -3x \qquad h(x) = 5$$

then:

$$\begin{aligned} k(x) &= f(x) + g(x) + h(x) \\ &= 2x^2 - 3x + 5 \end{aligned}$$

Now we can just take the derivatives of $f(x)$, $g(x)$ and $h(x)$ and then add up the individual derivatives to find $k'(x)$. **After all, the change in a sum is equal to the sum of the changes.**

$$\frac{d\,k(x)}{d\,x} = \frac{d\,f(x)}{d\,x} + \frac{d\,g(x)}{d\,x} + \frac{d\,h(x)}{d\,x}$$

$$k'(x) = f'(x) + g'(x) + h'(x)$$

$$k'(x) = 2 \cdot 2x^{2-1} - 1 \cdot 3x^{1-1} + 0 = 4x - 3$$

Example #1 – Total Revenue and Marginal Revenue

Total Revenue, denoted TR, is a function of the quantity of output that a firm produces, denoted Q, and the price at which the firm sells its output, denoted p. Specifically, Total Revenue is equal to the amount of output that a firm sells times the price. For example, if the firm sells 20 widgets at a price of $5 each, then its Total Revenue is $100.

If a firm is in a perfectly competitive market, then the firm cannot sell its output at a price higher than the one that prevails in the market (otherwise everyone would buy the products of competitor firms). So we can assume that the price is **constant**.

So what is a firm's Marginal Revenue? It's Marginal Revenue, denoted MR, is the derivative of Total Revenue with respect to a change in the quantity of output that the firm produces.

$$TR(Q) = p \cdot Q \quad \rightarrow \quad MR = \frac{d\,TR(Q)}{d\,Q} = p$$

Example #2 – Total Product and Marginal Product of Labor

If a firm produces output using "capital" – a fancy word for machinery – and labor, then the quantity of output that it produces – i.e. its Total Product, denoted by TP – is a function of two variables: capital, denoted by K, and labor, denoted by L.

$$TP(K,L) = K^{0.3} \cdot L^{0.7}$$

So what is the Marginal Product of Labor, denoted MPL? Marginal Product of Labor is the change in Total Product caused by an increase in Labor input. Marginal Product of Labor is the derivative of Total Product with respect to Labor.

Notice that we're looking solely at the change in Total Product that occurs when we vary the Labor input. We're not changing the capital stock, so when we take the derivative of Total Product with respect to Labor, we'll hold the firm's capital stock is fixed – i.e. we'll hold it **constant**.

$$TP(K,L) = K^{0.3} \cdot L^{0.7} \quad \rightarrow \quad MPL = \frac{d\,TP(K,L)}{d\,L} = 0.7 \cdot K^{0.3} \cdot L^{-0.3} = 0.7 \cdot \left(\frac{K}{L}\right)^{0.3}$$

Homework #1C

1. Find the derivative of each of the following functions:

a. $g(x) = 7x^6$

b. $k(y) = 3y^{-1}$

c. $m(q) = \frac{3}{2}q^{-2/3}$

d. $h(w) = -aw^2 + bw + \frac{c}{w}$

e. $u(z) = 5$

f. $y(x) = mx + b$

2. The Total Product of a firm, denoted by TP, depends on the amount of capital and labor that it employs. Denote capital by K and denote labor by L.

The Total Product function is given by: $TP(K, L) = K^{0.5} \cdot L^{0.5}$.

Throughout this problem, assume that the firm's capital stock is fixed at one unit.

a. Plot the Total Product function from zero units of Labor to four units of Labor. (Hint: Use graph paper if you have it).

b. Now find the Marginal Product of Labor by taking the derivative of the Total Product function with respect to Labor.

c. Plot the Marginal Product of Labor from zero units of Labor to four units of Labor.

3. Plot each of the following functions. Then find the derivative of each function and plot the derivative directly underneath your plot of the original function.

a. $f(x) = x^{1.5}$

b. $g(x) = x^{-0.5}$

If you plot the functions correctly, you will notice that the height of the plotted derivative is higher when the slope of the original function is steeper. Conversely, the height of the plotted derivative is lower when the slope of the original function is shallower.

4. The Total Cost function of a firm depends on the quantity of output that it produces, denoted by Q.

The Total Cost function is given by: $TC(Q) = Q^3 - 6Q^2 + 18Q + 6$.

a. Plot the Total Cost function from zero units of output to five units of output. (Hint: Use graph paper if you have it).

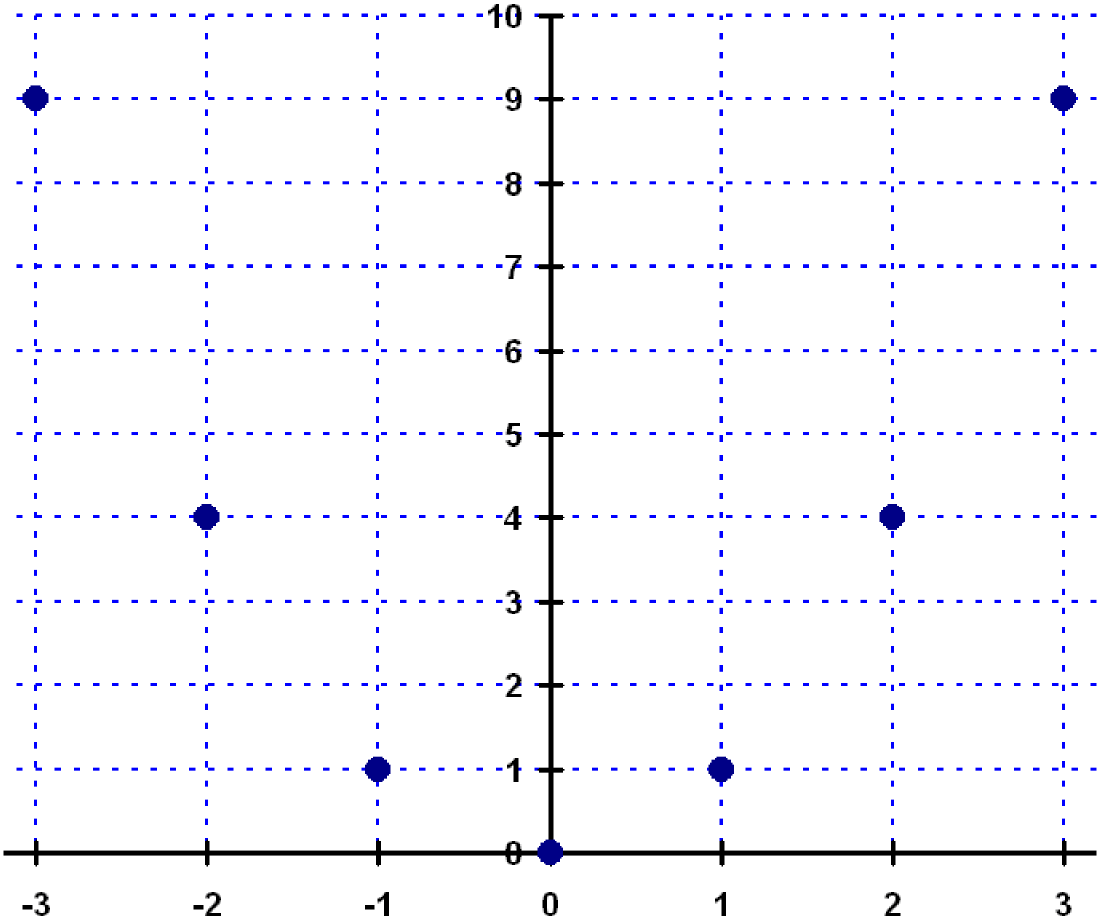
b. Does the Total Cost function ever slope downward? Or is it everywhere increasing?

c. Now find the Marginal Cost function by taking the derivative of the Total Cost function with respect to the quantity of output that the firm produces.

d. Plot the Marginal Cost function from zero units of output to five units.

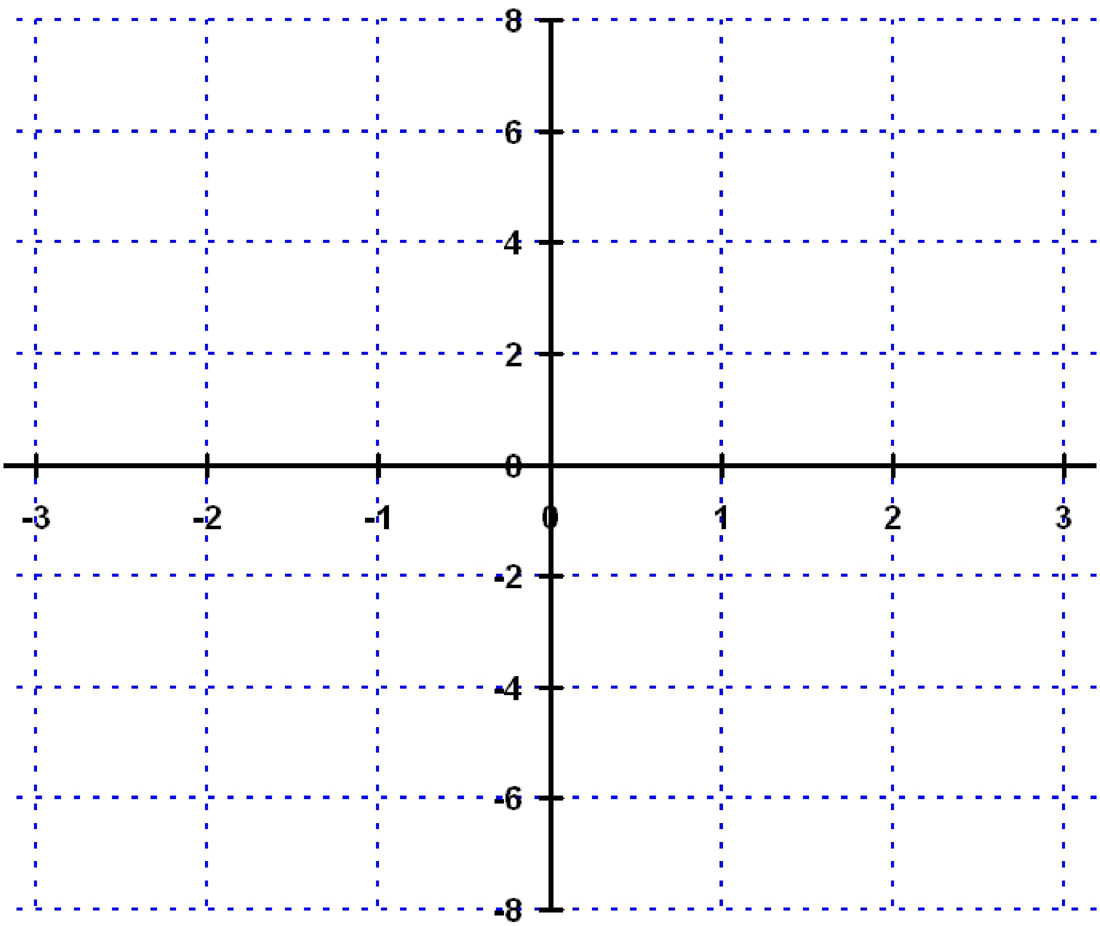
e. Does the Marginal Cost function ever slope downward? Or is it everywhere increasing?

f. If the Total Cost function never slopes downward, then why does the Marginal Cost function slope downward over some ranges of output?



$f(x) = x^2$

x	f(x)	$\frac{\Delta f(x)}{\Delta x}$
−3.0	9	
−2.5		
−2.0	4	
−1.5		
−1.0	1	
−0.5		
0.0	0	
0.5		
1.0	1	
1.5		
2.0	4	
2.5		
3.0	9	



$f'(x) = 2x$

x	$\frac{\Delta f(x)}{\Delta x}$	f′(x)
−3.0		
−2.5		
−2.0		
−1.5		
−1.0		
−0.5		
0.0		
0.5		
1.0		
1.5		
2.0		
2.5		
3.0		

Calculus Tricks #2

This set of calculus tricks explains the chain rule and the product-quotient rule. For the purposes of this course, our only need for these rules will be to show that:

- The percentage change in a product of two variables is equal to the sum of the percentage changes in each of the two variables.
- The percentage change in the ratio of two variables is equal to the percentage change in the numerator minus the percentage change in the denominator.

For example, if we're interested in the percentage change in Total Revenue, i.e. $TR = p \cdot Q$, then:

$$\frac{\Delta TR}{TR} = \frac{\Delta(p \cdot Q)}{(p \cdot Q)} = \frac{\Delta p}{p} + \frac{\Delta Q}{Q}$$

To take another example, if we're interested in the percentage change in GDP per capita, i.e. GDP/N (where N denotes population), then:

$$\frac{\Delta \text{GDP per capita}}{\text{GDP per capita}} = \frac{\Delta(GDP/N)}{(GDP/N)} = \frac{\Delta GDP}{GDP} - \frac{\Delta N}{N}$$

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the chain rule

Say you are considering a function that is a function of a function. That is:

$$h(x) = f(g(x))$$

In other words, the value of $h(x)$ changes as the function named "f" changes and the function named "f" changes as the function $g(x)$ changes.

To analyze this change, we can analyze a chain of causality that runs from x to $h(x)$.

$$x \rightarrow g(x) \rightarrow f(g(x)) = h(x)$$

So the derivative of $h(x)$ with respect to x is:

$$\frac{d\,h(x)}{d\,x} = \frac{d\,f(x)}{d\,g(x)} \cdot \frac{d\,g(x)}{d\,x}$$

which looks like the chain of causality flipped around:

$$h(x) = f(g(x)) \leftarrow g(x) \leftarrow x$$

So for example, if $g(x) = 3x + 1$ and if $f(g(x)) = (g(x))^2$, then $h(x) = (3x+1)^2$.

So there are two ways to take the derivative of $h(x)$ with respect to x. Using the methods you already learned, you could expand the terms in the function $h(x)$:

$$h(x) = (3x+1)^2 = 9x^2 + 6x + 1$$

and then take the derivative of $h(x)$ with respect to x, so that:

$$h'(x) = \frac{d\,h(x)}{d\,x} = 18x + 6$$

Expanding the terms of $(3x+1)^2$ can be rather tedious when you're working with a complicated function. Fortunately, the chain rule enables us to arrive at the same result, but in a somewhat quicker fashion:

$$\left.\begin{array}{lcr} f(g(x)) = (g(x))^2 & \Rightarrow & f'(g(x)) = 2\cdot g(x) \\ \\ g(x) = 3x+1 & \Rightarrow & g'(x) = 3 \end{array}\right\} \rightarrow \begin{aligned} h'(x) &= f'(g(x))\cdot g'(x) \\ &= 2\cdot g(x)\cdot 3 \\ &= 6\cdot(3x+1) \\ &= 18x+6 \end{aligned}$$

which yields exactly the same result as the one above.

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the product-quotient rule

Say you are considering a function that is the product of two functions, each of which is a function of the variable x. That is:

$$h(x) = f(x)\cdot g(x)$$

If we knew the explicit functional forms of $f(x)$ and $g(x)$, then we could multiply $f(x)$ by $g(x)$ and take the derivative of $h(x)$ with respect to x using the rules you already know. For example,

$$\text{if } f(x) = 3x \text{ and } g(x) = x^2, \text{ then } \begin{aligned} h(x) &= f(x)\cdot g(x) \\ &= 3x\cdot x^2 \\ &= 3x^3 \end{aligned} \text{ and } \frac{d\,h(x)}{d\,x} = h'(x) = \mathbf{9x^2}$$

But we can also consider the change in $h(x)$ as $f(x)$ changes holding $g(x)$ constant and the change in $h(x)$ as $g(x)$ changes holding $f(x)$ constant.

In other words: $$\frac{d\,h(x)}{d\,x} = \frac{d\,f(x)}{d\,x}\cdot g(x) + \frac{d\,g(x)}{d\,x}\cdot f(x) \quad \text{or} \quad h'(x) = f'(x)\cdot g(x) + g'(x)\cdot f(x)$$

Using the previous case where $f(x) = 3x$ and $g(x) = x^2$, we can write:

$$\begin{aligned} h'(x) &= f'(x)\cdot g(x) + g'(x)\cdot f(x) \\ &= 3\cdot x^2 + 2x\cdot 3x \\ &= 3x^2 + 6x^2 \\ &= 9x^2 \end{aligned}$$

which yields exactly the same result as the one above.

Now let's say you are considering a function that is a ratio of two functions, each of which is a function of the variable x. That is:

$$h(x) = \frac{f(x)}{g(x)} \quad \text{which can be rewritten as:} \quad h(x) = f(x) \cdot (g(x))^{-1}$$

To find the derivative of $h(x)$ with respect to x, we can perform the exact same analysis as we did in the previous example, but with the twist that we also have to use the chain rule on the term $(g(x))^{-1}$.

If we define a function $k(x)$ which is identically equal to $(g(x))^{-1}$, i.e. $k(x) \equiv (g(x))^{-1}$, then we can rewrite the function $h(x)$ as:

$$h(x) = f(x) \cdot k(x)$$

The derivative of $h(x)$ with respect to x is:

$$h'(x) = f'(x) \cdot k(x) + k'(x) \cdot f(x)$$

And the derivative of $k(x)$ with respect to x is:

$$\frac{d\,k(x)}{d\,x} = \frac{d\,(g(x))^{-1}}{d\,g(x)} \cdot \frac{d\,g(x)}{d\,x}$$

$$k'(x) = -1 \cdot (g(x))^{-2} \cdot g'(x) = -\frac{g'(x)}{(g(x))^2}$$

Plugging that into the derivative of $h(x)$ with respect to x:

$$h'(x) = f'(x) \cdot (g(x))^{-1} - \frac{g'(x)}{(g(x))^2} \cdot f(x)$$

$$h'(x) = \frac{f'(x)}{g(x)} - \frac{g'(x) \cdot f(x)}{(g(x))^2}$$

So let's consider: $h(x) = \frac{f(x)}{g(x)}$, where $f(x) = 6x^4 + 2x^2$ and $g(x) = 2x$. In such a case, $h(x) = 3x^3 + x$ and $h'(x) = \mathbf{9x^2 + 1}$. To illustrate the rule we just derived, let's use the rule to obtain the same result:

$$h'(x) = \frac{f'(x)}{g(x)} - \frac{g'(x) \cdot f(x)}{(g(x))^2}$$

$$\left. \begin{aligned} f(x) &= 6x \quad + 2x^2 \;\Rightarrow\; f'(x) = 24x^3 + 4x \\ g(x) &= 2x \qquad\quad \Rightarrow\; g'(x) = 2 \end{aligned} \right\} \Rightarrow \quad h'(x) = \frac{24x^3 + 4x}{2x} - \frac{2 \cdot (6x^4 + 2x^2)}{(2x)^2}$$

$$h'(x) = 12x^2 + 2 \quad - 3x^2 - 1 \qquad = \mathbf{9x^2 + 1}$$

Now, let's return to the original purpose of this set of Calculus Tricks, i.e. to show that:

- The percentage change in a product of two variables is equal to the sum of the percentage changes in each of the two variables.
- The percentage change in the ratio of two variables is equal to the percentage change in the numerator minus the percentage change in the denominator.

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Example #1 – a percentage change in Total Revenue

Once again Total Revenue is given by $TR = p \cdot Q$. Let's assume now that the price of output and the quantity of output produced evolve over time, so that $p = p(t)$ and $Q = Q(t)$, where "t" represents time. In such a case Total Revenue would also evolve over time $TR = TR(t)$.

So what's the percentage change in Total Revenue over time? First, we need to find the changes:

$$\frac{d\,TR(t)}{d\,t} = \frac{d\,p(t)\cdot Q(t)}{d\,t} = \frac{d\,p(t)}{d\,t}\cdot Q(t) + p(t)\cdot\frac{d\,Q(t)}{d\,t}$$

$$= TR'(t) \qquad = p'(t)\cdot Q(t) + p(t)\cdot Q'(t)$$

Since we're interested in a percentage change, we need to divide both sides by Total Revenue to get the percentage change in Total Revenue:

$$\frac{TR'(t)}{TR(t)} = \frac{p'(t)\cdot Q(t)}{p(t)\cdot Q(t)} + \frac{p(t)\cdot Q'(t)}{p(t)\cdot Q(t)}$$

$$\underset{TR}{\%\Delta} = \frac{p'(t)}{p(t)} + \frac{Q'(t)}{Q(t)} = \underset{\text{price}}{\%\Delta} + \underset{\text{quantity}}{\%\Delta}$$

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a note on time derivatives

When working with dynamic changes – that is: a change over time – economists usually denote a time derivative by placing a dot over the variable. I will frequently use this notation.

So for example, the derivative of price with respect to time would be denoted by $\dot{p}$

$$\frac{d\,p(t)}{d\,t} = p'(t) = \dot{p}$$

and the derivative of quantity with respect to time would be denoted by $\dot{Q}$

$$\frac{d\,Q(t)}{d\,t} = Q'(t) = \dot{Q}$$

(continued on the next page)

Example #2 – a percentage change in the Capital-Labor ratio

The Capital-Labor ratio – denoted: k – is defined as: $k \equiv \frac{K}{L}$, where K and L denotes capital and labor respectively.

Suppose that these two variables evolve over time so that: $K = K(t)$ and $L = L(t)$. This implies that the Capital-Labor ratio also evolves over time, so $k = k(t)$.

To avoid clutter, I'll drop the " t " from the functional notations.

So how does the Capital-Labor ratio evolve over time?

$$\frac{dk}{dt} = \frac{d}{dt} \cdot \left(K \cdot L^{-1}\right)$$

$$\dot{k} = L^{-1} \cdot \frac{dK}{dt} + K \cdot \frac{dL^{-1}}{dL} \cdot \frac{dL}{dt}$$

$$= \frac{\dot{K}}{L} - K \cdot \frac{\dot{L}}{L^2} = \frac{K}{L} \cdot \left(\frac{\dot{K}}{K} - \frac{\dot{L}}{L}\right)$$

Since $k \equiv \frac{K}{L}$, the derivation above implies that:

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}$$

The percentage change in the Capital-Labor ratio over time is equal to the percentage change in Capital over time minus the percentage change in Labor over time.

Some students have told me that they understand the product-quotient rule better when I explain the rules using difference equations.

Example #1 revisited – a percentage change in Total Revenue

Since Total Revenue is given by: $TR = p \cdot Q$, the percentage change in Total Revenue is:

$$\frac{\Delta TR}{TR} = \frac{\Delta(p \cdot Q)}{p \cdot Q} = \frac{p_2 \cdot Q_2 - p_1 \cdot Q_1}{p_1 \cdot Q_1}$$

where: p_1 is the inital price, Q_1 is the inital quantity, p_2 is the new price, Q_2 is the new quantity

Next, we're going to add a zero to the equation above. Adding zero leaves the value of the percentage change in Total Revenue unchanged.

We're going to add that zero in an unusual manner. The zero that we're going to add is:

$$0 = \frac{p_1 \cdot Q_2 - p_1 \cdot Q_2}{p_1 \cdot Q_1}$$

Adding our "unusual zero" yields:

$$\frac{\Delta TR}{TR} = \frac{p_2 \cdot Q_2 - p_1 \cdot Q_1}{p_1 \cdot Q_1} + \frac{p_1 \cdot Q_2 - p_1 \cdot Q_2}{p_1 \cdot Q_1}$$

Rearranging terms, we get:

$$\frac{\Delta TR}{TR} = \frac{(p_2 - p_1) \cdot Q_2}{p_1 \cdot Q_1} + \frac{p_1 \cdot (Q_2 - Q_1)}{p_1 \cdot Q_1}$$

Now notice that: $\Delta p = (p_2 - p_1)$ and $\Delta Q = (Q_2 - Q_1)$, therefore:

$$\frac{\Delta TR}{TR} = \frac{\Delta p}{p_1} \cdot \frac{Q_2}{Q_1} + \frac{\Delta Q}{Q_1}$$

Since we're considering very small changes: $\Delta Q \approx 0$, which implies that: $Q_2 \approx Q_1$ and $\frac{Q_2}{Q_1} \approx 1$.

Therefore we can write:

$$\frac{\Delta TR}{TR} = \frac{\Delta p}{p} + \frac{\Delta Q}{Q}$$

Example #2 revisited – a percentage change in the Capital-Labor ratio

Once again, define k as the Capital-Labor ratio, i.e.: $k \equiv \frac{K}{L}$, where K denote capital and L denotes labor. The percentage change in the Capital-Labor ratio is:

$$\frac{\Delta k}{k} = \frac{\Delta(K/L)}{K/L} = \frac{\frac{K_2}{L_2} - \frac{K_1}{L_1}}{K_1/L_1}$$

where: K_1 is the inital capital stock, L_1 is the inital labor force, K_2 is the new capital stock, L_2 is the new labor force

Once again, we're going to add an "unusual zero."

$$0 = \frac{\frac{K_1}{L_2} - \frac{K_1}{L_2}}{K_1/L_1}$$

Adding our "unusual zero" yields:

$$\frac{\Delta(K/L)}{K/L} = \frac{\frac{K_2}{L_2} - \frac{K_1}{L_1}}{K_1/L_1} + \frac{\frac{K_1}{L_2} - \frac{K_1}{L_2}}{K_1/L_1}$$

Rearranging terms, we get:

$$\frac{\Delta(K/L)}{K/L} = \frac{\frac{K_2}{L_2} - \frac{K_1}{L_2}}{K_1/L_1} + \frac{\frac{K_1}{L_2} - \frac{K_1}{L_1}}{K_1/L_1}$$

$$= \left(\frac{K_2}{L_2} - \frac{K_1}{L_2}\right) \cdot \frac{L_1}{K_1} + \left(\frac{K_1}{L_2} - \frac{K_1}{L_1}\right) \cdot \frac{L_1}{K_1}$$

$$= \left(\frac{K_2 - K_1}{K_1}\right) \cdot \frac{L_1}{L_2} + \left(\frac{L_1}{L_2} - \frac{L_1}{L_1}\right) \cdot \frac{K_1}{K_1}$$

$$= \frac{\Delta K}{K_1} \cdot \frac{L_1}{L_2} + \left(\frac{L_1}{L_2} - 1\right) \quad \leftarrow \text{note that} : 1 = \frac{L_2}{L_2}$$

$$= \frac{\Delta K}{K_1} \cdot \frac{L_1}{L_2} + \left(\frac{L_1 - L_2}{L_2}\right)$$

$$= \frac{\Delta K}{K_1} \cdot \frac{L_1}{L_2} - \frac{\Delta L}{L_2}$$

The derivation above uses the definitions: $\Delta K = K_2 - K_1$ and $\Delta L = L_2 - L_1$.

Since we're considering very small changes: $\Delta L \approx 0$, which implies that: $L_2 \approx L_1$ and $\frac{L_1}{L_2} \approx 1$.

Therefore we can write:

$$\frac{\Delta(K/L)}{K/L} = \frac{\Delta K}{K} - \frac{\Delta L}{L}$$

Homework #1D

1. Let $Y(t)$ denote output as a function of time and let $L(t)$ denote the labor force as a function of time.

a. What is the ratio of output per worker?

b. How does it evolve over time?

2. Let $Y(t)$ denote output as a function of time, let $L(t)$ denote the labor force as a function of time and let $A(t)$ denote a level labor efficiency, so that $A(t) \cdot L(t)$ is the "effective labor force."

a. What is the ratio of output per unit of effective labor?

b. How does it evolve over time?

3. Let $K(t)$ denote the capital stock as a function of time, let $L(t)$ denote the labor force as a function of time and let $A(t)$ denote a level labor efficiency, so that $A(t) \cdot L(t)$ is the "effective labor force." Let $\tilde{k}(t)$ denote the ratio of capital to effective labor.

a. What is the ratio of capital per unit of effective labor?

b. How does it evolve over time?

c. Find the derivative: $\frac{d\,\tilde{k}(t)^{\alpha}}{d\,t}$. Hint: Use the chain rule. It makes life a lot easier.

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Notes on Logarithms

When I initially designed this course, I did not plan to teach you how to use logarithms. Van den Berg's textbook however assumes that you understand logarithms, so I've written these notes to enable you to better understand the equations in his text.

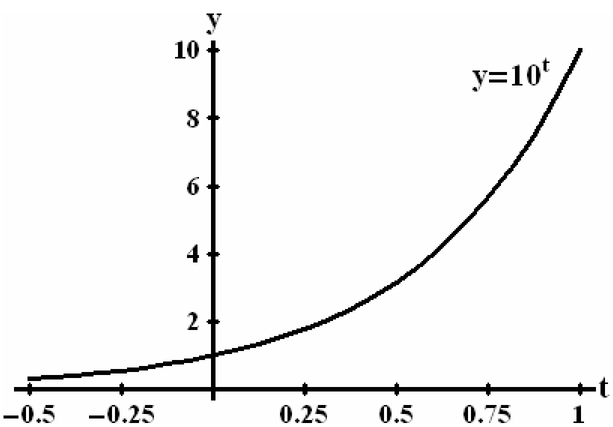
Logarithms start with a given base number. The base number can be any real number. The simplest base to use is 10, but the preferred base is the irrational number: $e = 2.71828...$. These notes explain the basic idea of logarithms using the base number 10. Then once you've grasped the basic idea behind logarithms, these notes will introduce the preferred base.

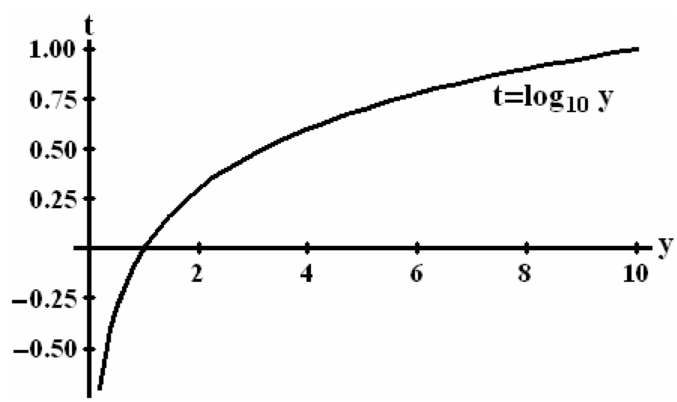
Now that we've temporarily chosen a base of 10, let's pick another number, say: 1000. The basic idea of logarithms is to answer the question: "10 raised to what power will equals 1000?" The answer of course is: "10 raised to the third power equals 1000." That is: $10^3 = 1000$. Mathematically, we say: "The logarithm of 1000 to the base of 10 equals 3." That is: $\log_{10} 1000 = 3$.

Now let's pick another number, say: 0.01 and once again ask: "10 raised to what power will equals 0.01?" The answer this time is: "10 raised to the power –2 equals 0.01." That is: $10^{-2} = 0.01$. Mathematically, we say: "The logarithm of 0.01 to the base of 10 equals –2." That is: $\log_{10} 0.01 = -2$.

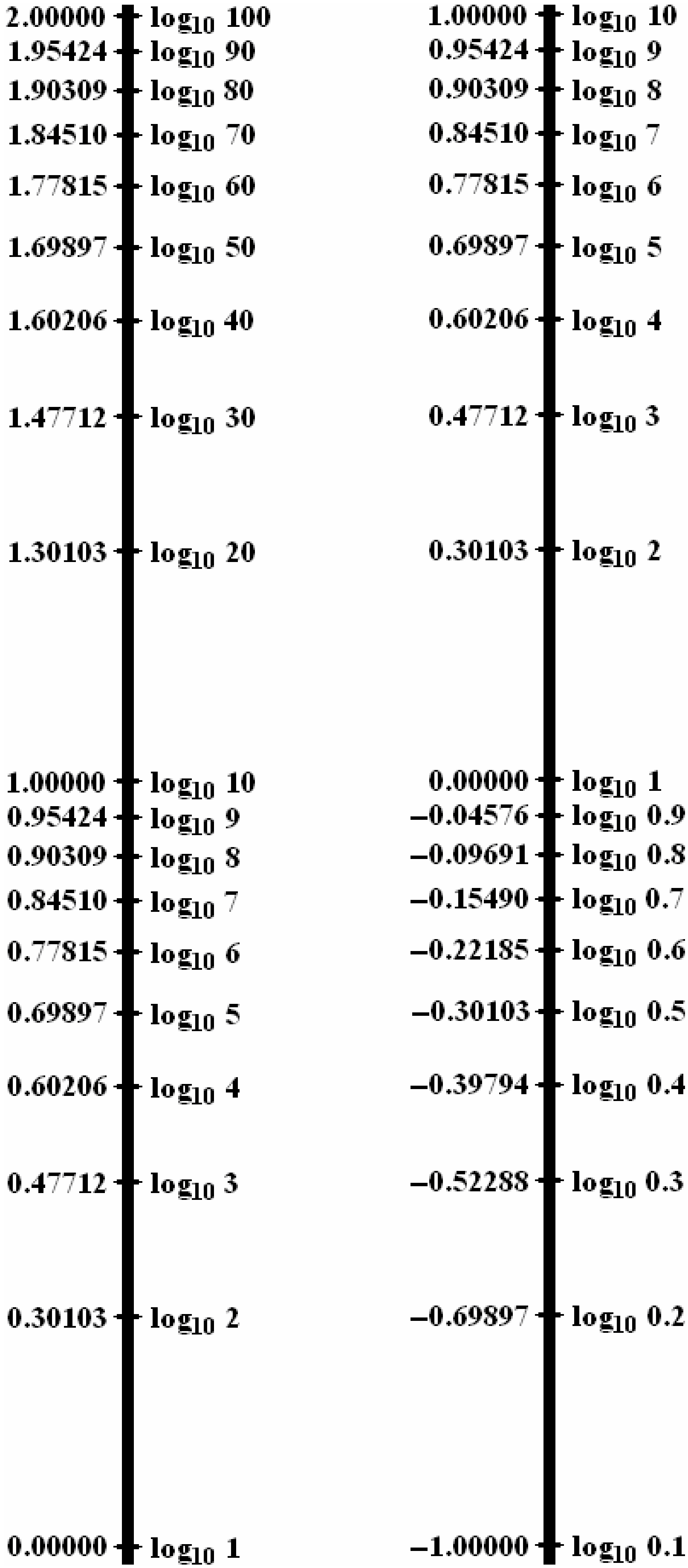
$10^3 = 1000$	$\log_{10} 1000 = 3$
$10^2 = 100$	$\log_{10} 100 = 2$
$10^1 = 10$	$\log_{10} 10 = 1$
$10^0 = 1$	$\log_{10} 1 = 0$
$10^{-1} = 0.1$	$\log_{10} 0.1 = -1$
$10^{-2} = 0.01$	$\log_{10} 0.01 = -2$
$10^{-3} = 0.001$	$\log_{10} 0.001 = -3$

This relationship is summarized in the table above and is depicted in the graphs below.





It should also be intuitively clear that if we had chosen a different base number, say: 4, then we could ask the question: "4 raised to what power equals 16?" The answer this time is: "4 raised to the second power equals 16." That is: $4^2 = 16$. Mathematically, we say: "The logarithm of 16 to the base of 4 equals 2." That is: $\log_4 16 = 2$.



Logarithms are useful because they allow us to perform the mathematical operations of multiplication and division using the simpler operations of addition and subtraction.

For example, you already know that: $2 \times 4 = 8$, so look at the logarithmic scales at left and observe that:

$$\begin{array}{rr} \log_{10} 2 & 0.30103 \\ +\log_{10} 4 & +0.60206 \\ \hline \log_{10} 8 & 0.90309 \end{array}$$

Similarly, you know that: $\frac{40}{8} = 5$.
Looking again at the logarithmic scales, you can see that:

$$\begin{array}{rr} \log_{10} 40 & 1.60206 \\ -\log_{10} 8 & -0.90309 \\ \hline \log_{10} 5 & 0.69897 \end{array}$$

In fact, before technology enabled us all to carry a calculator our pocket, people performed multiplication and division using slide rules that had base 10 logarithmic scales.

So why does this "trick" work? To answer this question, first recall that:

$$10^2 \cdot 10^3 = 10^5$$
$$100 \cdot 1000 = 100{,}000$$

$$10^2 \cdot 10^{-3} = 10^{-1}$$
$$\frac{100}{1000} = 0.1$$

So the "trick" works because the numerical value of a logarithm is an exponent and because you can add (or subtract) exponents in a multiplication problem (or division problem) so long as the exponents are the powers of a common base number.

On the previous page, we established two rules of logarithms:

$$\text{Rule I:} \qquad \log_{10}(a \cdot b) = \log_{10} a + \log_{10} b$$
$$\text{Rule II:} \qquad \log_{10}\left(\frac{a}{b}\right) = \log_{10} a - \log_{10} b$$

We can use Rule I to establish yet another rule:

$$\text{Rule III:} \qquad \log_{10}\left(a^c\right) = c \cdot \log_{10} a$$

For example: $4^3 = 4 \cdot 4 \cdot 4$, therefore:

$$\begin{aligned} \log_{10}\left(4^3\right) &= \log_{10}(4 \cdot 4 \cdot 4) \\ &= \log_{10} 4 + \log_{10} 4 + \log_{10} 4 \\ &= 3 \cdot \log_{10} 4 \end{aligned}$$

Of course, **the rules above apply to logarithms to all bases.** After all, the numerical value of a logarithm is just an exponent and an exponent can be attached to any base number.

We've been working with logarithms to the base of 10, but in analytical work the preferred base is the irrational number: $e = 2.71828...$. Logarithms to the base of e are called **natural logarithms** (abbreviated "ln"): $\log_e a \equiv \ln a$. The rules of natural logarithms are the same as the ones derived above:

$$\text{Rule I:} \qquad \ln(a \cdot b) = \ln a + \ln b$$
$$\text{Rule II:} \qquad \ln\left(\frac{a}{b}\right) = \ln a - \ln b$$
$$\text{Rule III:} \qquad \ln\left(a^c\right) = c \cdot \ln a$$

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pitfalls to avoid

Finally, there are two pitfalls to avoid.

First, observe from Rule I that $\ln(a + b)$ is NOT equal to $\ln a + \ln b$. Similarly, Rule II tells us that $\ln(a - b)$ is NOT equal to $\ln a - \ln b$.

Second, logarithms of non-positive numbers are undefined. For example, in the graphs on the first page, we used the equation $y = 10^t$ to obtain the relationship $t = \log_{10} y$. Therefore if $y = 0$, then the value of t must be negative infinity.

So what would the value of t be if y were a negative number? … That's a trick question. If y were a negative number, then t could not possibly be a real number. For this reason, logarithms of negative numbers are undefined.