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Math Methods 7025X – Problem Set

I have designed the following set of problems to help you put the mathematical techniques that you are learning into practice. This course is intended to support the other economics courses in the graduate program, so I have written two micro problems, two macro problems and two econometrics problems. We will discuss these problems in class over the course of the semester.

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Problem #1 – Consider a simple model of national income in three variables: Y (national income), C (consumption) and G (government spending). In the model, investment, I_0, and taxation, T_0, are exogenously determined and government spending has a counter-cyclical component: G increases as Y decreases.

$$\begin{array}{lll} Y = C + I_0 + G & & \\ C = \alpha + \beta \cdot (Y - T_0) & 0 < \alpha & 0 < \beta < 1 \\ G = \gamma - \eta \cdot Y & 0 < \gamma & 0 < \eta < 1 \end{array}$$

1. Set up the system of equations in matrix form.
2. Solve the system of equations to obtain the equilibrium values of Y, C and G.
3. Obtain the equilibrium budget balance (i.e. the equilibrium value of: $T_0 - G$).
4. What is the effect of an increase in taxes on the budget balance? Discuss.

Matrix Cheat Sheet

Given the 3×3 matrix A:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

the determinant of A is:

$$\begin{aligned} |A| = \; & a_{11} \cdot a_{22} \cdot a_{33} + a_{21} \cdot a_{32} \cdot a_{13} + a_{31} \cdot a_{12} \cdot a_{23} \\ - \; & a_{13} \cdot a_{22} \cdot a_{31} - a_{23} \cdot a_{32} \cdot a_{11} - a_{33} \cdot a_{12} \cdot a_{21} \end{aligned}$$

and the inverse of A is:

$$A^{-1} = \frac{1}{|A|} \cdot \begin{bmatrix} (a_{22} \cdot a_{33} - a_{23} \cdot a_{32}) & (a_{13} \cdot a_{32} - a_{12} \cdot a_{33}) & (a_{12} \cdot a_{23} - a_{13} \cdot a_{22}) \\ (a_{23} \cdot a_{31} - a_{21} \cdot a_{33}) & (a_{11} \cdot a_{33} - a_{13} \cdot a_{31}) & (a_{13} \cdot a_{21} - a_{11} \cdot a_{23}) \\ (a_{21} \cdot a_{32} - a_{22} \cdot a_{31}) & (a_{12} \cdot a_{31} - a_{11} \cdot a_{32}) & (a_{11} \cdot a_{22} - a_{12} \cdot a_{21}) \end{bmatrix}.$$

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Problem #2 – Output per worker, y, is a function of capital per worker, k, which exhibits diminishing marginal returns. The evolution of capital per worker over time, $\dot{k} \equiv dk/dt$, is an increasing function of the saving rate, s, and a decreasing function of the depreciation rate, δ, and the labor force growth rate, n.

$$\begin{array}{llll} y = k^{\alpha} & 0 < \alpha < 1 & y \equiv y(t) & k \equiv k(t) \\ \dot{k} = s \cdot k^{\alpha} - (\delta + n) \cdot k & 0 < s < 1 & 0 < \delta < 1 & 0 < n < 1 \end{array}$$

1. Solve for the steady-state level of capital per worker, k_{ss} (i.e. the value of k when $\dot{k} = 0$).
2. Derive the marginal product of capital (i.e. $MPK \equiv dy/dk$).
3. Use the chain rule to derive the growth rate of output per worker (i.e. $\dot{y}/y$).
4. Why does the growth rate of output per worker depend on the marginal product of capital? Discuss.

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Problem #3 – A mortgage lender seeks to maximize the expected value of its portfolio. The portfolio, of course, is the sum of all of the mortgages in it, so no generality is lost by examining the case of one loan:

$$E[port] = (1 - p) \cdot B + p \cdot \gamma \cdot B$$

where:
- $E[port]$ is the expected value of the portfolio
- B is the principal balance
- p is the probability of foreclosure
- γ is the percentage of the principal balance recovered in a foreclosure sale

By definition: $0 \leq \gamma \leq 1$. In other words, lenders may not recover more than the principal balance through the foreclosure process. In cases where the borrower owes more than the home is worth (i.e. the borrower is "underwater"), lenders recover far less than the principal balance.

Define the amount recovered as:

$$\gamma \cdot B \equiv V - L$$

where:
- V is the sale price at foreclosure
- L is the legal fees incurred by foreclosure

Further, assume that the borrower's probability of foreclosure is an increasing function of his/her "balance-to-value ratio" (i.e. B/V):

$$p \equiv p\left(\frac{B}{V}\right) \qquad p' > 0$$

In other words, borrowers who are deeper underwater are more likely to enter the foreclosure process. In such cases, reducing principal balances would reduce foreclosure-related losses (by reducing the probability of foreclosure). On the other hand, principal balance reductions are a direct loss for the lender.

The following questions ask you to determine the conditions under which lenders have (or do not have) an interest in reducing principal balances. In other words, what are the conditions under which a lender would increase the expected value of its portfolio by reducing principal balances?

1. Derive the marginal benefit of reducing principal balances.
2. Derive the marginal cost of reducing principal balances.
3. What is the necessary condition for maximizing $E[port]$ with respect to the principal balance?
4. What is the sufficient condition for maximizing $E[port]$?
5. Explain why a lender has little interest in principal balance reduction when it can recover a large percentage of the principal balance through the foreclosure process. Explain why principal balance reductions might increase the expected value of the lender's portfolio when the percentage that the lender can recover is small.

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Problem #4 – The following questions ask you to derive the least squares estimates of regression coefficients. Consider the following regression model:

$$y_i = \alpha + \beta x_i + \varepsilon_i$$

where ε_i is the error term. Rearranging terms, squaring both sides and summing over observations yields the sum of squared errors:

$$\sum \varepsilon_i^2 = \sum (y_i - \alpha - \beta x_i)^2$$

The estimates of regression coefficients are the values of α and β that minimize the sum of squared errors.

1. Obtain the first-order conditions for a minimum by taking the partial derivative of $\sum \varepsilon_i^2$
   (a) with respect to α
   (b) with respect to β

Define the mean of x as: $\bar{x} \equiv \frac{1}{N}\sum x_i$ and define the mean of y as: $\bar{y} \equiv \frac{1}{N}\sum y_i$.

2. Show that the first-order conditions imply that: $\widehat{\alpha} = \bar{y} - \widehat{\beta}\bar{x}$.
3. Show that the first-order conditions imply that: $\widehat{\beta} = \dfrac{\frac{1}{N}\sum x_i y_i - \bar{x}\bar{y}}{\frac{1}{N}\sum x_i^2 - \bar{x}^2}$.
4. Rearrange terms to show that: $\widehat{\beta} = \dfrac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$.

Next, you need to show that the second-order conditions for a minimum are satisfied.

5. Take the second partial derivative of $\sum \varepsilon_i^2$
   (a) with respect to α
   (b) with respect to β
   (c) with respect to α and β (i.e. the "cross-partial")

6. Show that the second partial with respect to α is positive.

7. Show that the second partial with respect to β is positive.

8. Set up the Hessian matrix and show that its determinant is positive. (**Hint:** If you rearrange terms, you'll see that the determinant is a function of the variance of x).

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Problem #5 – The following questions ask you to derive the maximum likelihood estimates of the mean and variance. Assuming that x is distributed normally with mean μ and variance σ^2, the likelihood function can be written as the product of the probability densities for each observation:

$$\pounds(\mu, \sigma) = \prod \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2\right)$$

1. Take the natural logarithm of the likelihood function to obtain the log-likelihood function.

2. Obtain the first-order conditions for a maximum by taking the partial derivative of the log-likelihood function:

   (a) with respect to μ

   (b) with respect to σ

3. Show that the first-order conditions imply that: $\widehat{\mu} = \frac{1}{N}\sum x_i$.

4. Show that the first-order conditions imply that: $\widehat{\sigma}^2 = \frac{1}{N}\sum (x_i - \mu)^2$.

Next, you need to show that the second-order conditions for a maximum are satisfied.

5. Take the second partial derivative of the log-likelihood function:

   (a) with respect to μ

   (b) with respect to σ

   (c) with respect to μ and σ (i.e. the "cross-partial")

6. Show that the second partial with respect to μ is negative.

7. Show that the second partial with respect to σ is negative.

8. Set up the Hessian matrix and show that its determinant is positive. (**Hint:** Make use of the fact that $\widehat{\mu} = \frac{1}{N}\sum x_i$ when the first-order conditions are satisfied).

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Problem #6 – Julia is a Brooklyn College student. Her utility depends on her consumption of two goods: hamburgers, h, and marijuana, m:

$$U = u(h, m)$$

To fund her consumption habits, Julia works at a bank and receives an hourly wage, w. She is expected to work a fixed number of hours per week, L, but when she consumes marijuana she arrives at work late the next day and is unable to make up the hours that she missed (resulting in a loss of income).

Specifically, the number of hours that she misses is equal to the square of the number of joints of marijuana that she smokes. In other words, if she smokes one joint she misses one hour of work. If she smokes two joints, she misses four hours of work. If she smokes three joints, she misses nine hours of work, etc. So her budget constraint can be written as:

$$w \cdot \left(L - m^2\right) = p_h \cdot h + p_m \cdot m$$

where p_h is the price of a hamburger and p_m is the price of one marijuana joint.

1. Set up the Lagrangian for the purpose of maximizing Julia's utility subject to her budget constraint.
2. Obtain the first-order conditions for a maximum by taking the partial derivative of the Lagrangian:
   (a) with respect to hamburgers, h
   (b) with respect to marijuana, m
3. Solve each of the first-order conditions for the Lagrange multiplier.
4. Set the resulting equations equal to each other and rearrange terms so that the marginal rate of substitution of hamburgers for marijuana is set equal to the effective relative price of marijuana.
5. What does the term on each side of the resulting equation represent?

Now suppose that Julia's utility function is given by:

$$u(h, m) = h^{\alpha} \cdot m^{1-\alpha}.$$

6. Take the partial derivatives of Julia's utility function and plug them into the marginal rate of substitution.
7. Solve the budget constraint for hamburgers and insert the result into the marginal rate of substitution.
8. Once again, set the marginal rate of substitution equal to the effective relative price of marijuana. Then use the quadratic formula to solve the resulting equation for Julia's utility maximizing consumption of marijuana.

Suppose that: $\alpha = \frac{4}{7}$, Julia's wage is \$5 per hour, she is expected to work 24 hours per week, the price of a hamburger is \$5 per hamburger and the price of a marijuana joint is \$10 per joint.

9. Show that Julia maximizes her utility by consuming 16 hamburgers and 2 marijuana joints.