Calculus Tricks

Calculus is <u>not</u> a pre-requisite for this course. However, the foundations of economics are based on calculus, so what we'll be discussing over the course of the semester is the intuition behind models constructed using calculus.

It's not surprising therefore that the students who do better in economics courses are the ones who have a better understanding of calculus – **even when calculus is not a required part of the course**. So if you want to do well in this course, you should learn a little calculus.

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Many times throughout the course, we'll be discussing marginalism – e.g. marginal cost, marginal revenue, marginal product of labor, marginal product of capital, marginal utility, marginal rate of substitution, marginal rate of transformation, etc.

Whenever you see "marginal …" it means "the derivative of …"

A derivative is just a slope. So, for example, let's say **labor is used to produce output**

- if TP stands for Total Production (quantity produced),
- if L stands for Labor input and
- if Δ denotes a change,

then if I write: $\frac{\Delta TP}{\Delta L}$ that's the change in Total Production divided by the change in Labor.

- It's the slope of the total production function.
- It's the derivative of the production function with respect to labor input.
- It's the marginal product of labor (MPL).

So if you understand derivatives, you'll understand the course material much better.

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a few preliminaries – exponents

You should recall from your high school algebra classes that when you see an exponent, it simply means multiply the number by itself the number of times indicated by the exponent.

$$x^3 = x \cdot x \cdot x$$

Now if you divide both sides of the above equation by x:

$$\frac{x^3}{x} = \frac{x \cdot x \cdot x}{x} = x^2$$

But what if you see the something like: x^0? Well, that's simply equal to:

$$x^0 = \frac{x^1}{x} = \frac{x}{x} = 1$$

$$
\begin{aligned}
2^3 &= 2\cdot 2\cdot 2 = 8 = \frac{16}{2} = \frac{2^4}{2} \\
2^2 &= 2\cdot 2 = 4 = \frac{8}{2} = \frac{2^3}{2} \\
2^1 &= 2 = 2 = \frac{4}{2} = \frac{2^2}{2} \\
2^0 &= 1 = 1 = \frac{2}{2} = \frac{2^1}{2} \\
2^{-1} &= \frac{1}{2} = \frac{1}{2} = \frac{1}{2} = \frac{2^0}{2} \\
2^{-2} &= \frac{1}{2\cdot 2} = \frac{1}{4} = \frac{1/2}{2} = \frac{2^{-1}}{2} \\
2^{-3} &= \frac{1}{2\cdot 2\cdot 2} = \frac{1}{8} = \frac{1/4}{2} = \frac{2^{-2}}{2}
\end{aligned}
$$

Similarly, $x^{-1} = \frac{x^0}{x} = \frac{1}{x}$ and $x^{-2} = \frac{x^{-1}}{x} = \frac{1/x}{x} = \frac{1}{x \cdot x} = \frac{1}{x^2}$.

But what about $x^{0.5}$? That's the square root of x: $x^{0.5} = \sqrt{x}$. Ex. $16^{0.5} = \sqrt{16} = 4$

By the same logic as before: $x^{-0.5} = \frac{1}{\sqrt{x}}$. Ex. $9^{-0.5} = \frac{1}{\sqrt{9}} = \frac{1}{3}$

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a few preliminaries – functions

You may have seen something like this in your high school algebra classes: $f(x)$. This notation means that there is a function named "f" whose value depends on the value of the variable called "x."

Some examples of functions in economics include:

- The quantity of output that a firm produces depends on the amount of labor that it employs. In such a case, we can define a function called "TP" (which stands for Total Production) whose value depends on a variable called "L" (which stands for Labor). So we would write: $TP(L)$.
- A firm's total cost of producing output depends on the amount of output that it produces. In such a case, we can define a function called "TC" (which stands for Total Cost) whose value depends on a variable called "Q" (which stands for Quantity). So we would write: $TC(Q)$.
- A firm's total revenue from selling output depends on the amount of output that it produces. In such a case, we can define a function called "TR" (which stands for Total Revenue) whose value depends on a variable called "Q" (which stands for Quantity). So we would write: $TR(Q)$.

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derivatives

Now let's return to the original purpose of these notes – to show you how to take a derivative.

A derivative is the slope of a function. For those of you who saw $f(x)$ in your high school algebra classes, you may recall taking a derivative called "f-prime of x," $f'(x)$.

What you were doing was you were finding the slope of the function $f(x)$. You were finding how much the value of the function $f(x)$ changes as x changes.

x	$f(x)$	$\frac{\Delta f(x)}{\Delta x}$	true $f'(x)$
0	0	−−	0
1	3	3	6
2	12	9	12
3	27	15	18

So let's define the function: $f(x) = 3x^2$ and let's look at how the value of $f(x)$ changes as we increase x by one unit increments. Once again, let Δ denote a change.

The third column is our rough measure of the slope. The fourth column – entitled true $f'(x)$ – is the true measure of the slope of $f(x)$ evaluated at each value of x. The values differ greatly between the two columns because we are looking at "large" changes in x as opposed to the infinitesimally small changes described in the notes entitled: "What's the Difference between Marginal Cost and Average Cost?"

Why does it make a difference whether we look at small or large changes? Consider the following derivation of the slope of $f(x)$:

$$f'(x) = \frac{\Delta f(x)}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{3(x+\Delta x)^2 - 3x^2}{\Delta x} = \frac{3(x+\Delta x)\cdot(x+\Delta x) - 3x^2}{\Delta x} = \frac{3\left(x^2 + 2x\Delta x + (\Delta x)^2\right) - 3x^2}{\Delta x}$$

$$= \frac{3x^2 + 6x\Delta x + 3(\Delta x)^2 - 3x^2}{\Delta x} = \frac{6x\Delta x + 3(\Delta x)^2}{\Delta x} = \frac{6x\Delta x}{\Delta x} + \frac{3(\Delta x)^2}{\Delta x}$$

$$f'(x) = 6x + 3\Delta x$$

If we look at one unit changes in the value of x – i.e. $\Delta x = 1$ – then the slope of $f(x)$ evaluated at each value of x is equal to $6x + 3\Delta x$ which equals $6x + 3$ since $\Delta x = 1$.

If we look at changes in x that are so small that the changes are approximately zero – i.e.: $\Delta x \approx 0$ – then the slope of $f(x)$ evaluated at each value of x is approximately equal to $6x$ and gets closer and closer to $6x$ as the change in x goes to zero.

So if $f(x) = 3x^2$, then $f'(x) = 6x$.

Since we'll be looking at infinitesimally small changes in x, we'll stop using the symbol Δ to denote a change and start using the letter d to denote an infinitesimally small change.

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calculus tricks – an easy way to find derivatives

For the purposes of this course, there are only three calculus rules you'll need to know:

- the constant-function rule
- the power-function rule and
- the sum-difference rule.

the constant-function rule

If $f(x) = 3$, then the value of $f(x)$ doesn't change x as changes – i.e. $f(x)$ is constant and equal to 3.

So what's the slope? Zero. Why? Because a change in the value of x doesn't change the value of $f(x)$. In other words, the change in the value of $f(x)$ is zero.

So if $f(x) = 3$, then $\frac{d\,f(x)}{d\,x} = f'(x) = 0$.

the power-function rule

Now if the value of x in the function $f(x)$ is raised to a power (i.e. it has an exponent), then all we have to do to find the derivative is "roll the exponent over."

To roll the exponent over, multiply the original function by the original exponent and subtract one from the original exponent. For example:

$$f(x) = 5x^3 \qquad \frac{d\,f(x)}{d\,x} = f'(x) = 15x^2$$

$$5x^3 \;\rightarrow\; 3\cdot 5x^{3-1} \;=\; 15x^2$$

$$g(x) = 4x^{1/2} = 4\sqrt{x} \qquad \frac{d\,g(x)}{d\,x} = g'(x) = 2x^{-1/2} = \frac{2}{\sqrt{x}}$$

$$4x^{1/2} \;\rightarrow\; \frac{1}{2}\cdot 4x^{\frac{1}{2}-1} \;=\; 2x^{-1/2}$$

the sum-difference rule

Now, say the function you are considering contains the variable x in two or more terms.

$$k(x) = 2x^2 - 3x + 5$$

if we define:

$$f(x) = 2x^2 \qquad g(x) = -3x^1 = -3x \qquad h(x) = 5$$

then:

$$\begin{aligned} k(x) &= f(x) + g(x) + h(x) \\ &= 2x^2 - 3x + 5 \end{aligned}$$

Now we can just take the derivatives of $f(x)$, $g(x)$ and $h(x)$ and then add up the individual derivatives to find $k'(x)$. **After all, the change in a sum is equal to the sum of the changes.**

$$\frac{d\,k(x)}{d\,x} = \frac{d\,f(x)}{d\,x} + \frac{d\,g(x)}{d\,x} + \frac{d\,h(x)}{d\,x}$$

$$k'(x) = f'(x) + g'(x) + h'(x)$$

$$k'(x) = 2\cdot 2x^{2-1} - 1\cdot 3x^{1-1} + 0 = 4x - 3$$

Example #1 – Total Revenue and Marginal Revenue

Total Revenue, denoted TR, is a function of the quantity of output that a firm produces, denoted Q, and the price at which the firm sells its output, denoted p. Specifically, Total Revenue is equal to the amount of output that a firm sells times the price. For example, if the firm sells 20 widgets at a price of $5 each, then its Total Revenue is $100.

If a firm is in a perfectly competitive market, then the firm cannot sell its output at a price higher than the one that prevails in the market (otherwise everyone would buy the products of competitor firms). So we can assume that the price is **constant**.

So what is a firm's Marginal Revenue? It's Marginal Revenue, denoted MR, is the derivative of Total Revenue with respect to a change in the quantity of output that the firm produces.

$$TR(Q) = p \cdot Q \quad \rightarrow \quad MR = \frac{d\,TR(Q)}{d\,Q} = p$$

Example #2 – Total Product and Marginal Product of Labor

If a firm produces output using "capital" – a fancy word for machinery – and labor, then the quantity of output that it produces – i.e. its Total Product, denoted by TP – is a function of two variables: capital, denoted by K, and labor, denoted by L.

$$TP(K,L) = K^{0.5} \cdot L^{0.5}$$

So what is the Marginal Product of Labor, denoted MPL? Marginal Product of Labor is the change in Total Product caused by an increase in Labor input. Marginal Product of Labor is the derivative of Total Product with respect to Labor.

Notice that we're looking solely at the change in Total Product that occurs when we vary the Labor input. We're not changing the capital stock, so when we take the derivative of Total Product with respect to Labor, we'll hold the firm's capital stock is fixed – i.e. we'll hold it **constant**.

$$TP(K,L) = K^{0.5} \cdot L^{0.5} \quad \rightarrow \quad MPL = \frac{d\,TP(K,L)}{d\,L} = 0.5 \cdot K^{0.5} \cdot L^{-0.5}$$